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Sensitivity in risk analyses with uncertain numbers

Scott Ferson and W. Troy Tucker
Applied Biomathematics
Setauket, New York 11733

Abstract

Sensitivity analysis is a study of how changes in the inputs to a model influence the results of the model. Many techniques have recently been proposed for use when the model is probabilistic. This report considers the related problem of sensitivity analysis when the model includes uncertain numbers that can involve both aleatory and epistemic uncertainty and the method of calculation is Dempster-Shafer evidence theory or probability bounds analysis. Some traditional methods for sensitivity analysis generalize directly for use with uncertain numbers, but, in some respects, sensitivity analysis for these analyses differs from traditional deterministic or probabilistic sensitivity analyses. A case study of a dike reliability assessment illustrates several methods of sensitivity analysis, including traditional probabilistic assessment, local derivatives, and a "pinching" strategy that hypothetically reduces the epistemic uncertainty or aleatory uncertainty, or both, in an input variable to estimate the reduction of uncertainty in the outputs. The prospects for applying the methods to black box models are also considered.

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Executive summary

Sensitivity analysis is the general term for systematic study of how changes in the inputs to a model would influence the results of the model. Sensitivity analyses are conducted for three fundamental reasons: (i) to understand the reliability or robustness of the conclusions and inferences drawn from an assessment, (ii) to identify how best to control a system or manage its outputs, and (iii) to focus future empirical studies so that effort might be expended to improve estimates of inputs that would lead to the most improvement in the estimates of the outputs. Sensitivity analysis is tightly intertwined with uncertainty analysis. Sensitivity analysis has many manifestations in probabilistic risk analysis and there are many disparate approaches based on various measures of sensitivity, influence and response. When probabilistic analyses are generalized to address both epistemic and aleatory uncertainty, new methods of calculation are needed such as Dempster-Shafer evidence theory (DST) and probability bounds analysis (PBA). The relationship between sensitivity analysis and these methods is subtle. The report makes the following conclusions:

1. DST and PBA are themselves a kind of sensitivity analysis. They can answer a wide variety of what-if questions about an analysis with considerable comprehensiveness.
2. Some traditional methods for parametric sensitivity analysis based on using derivatives as estimates of local sensitivities generalize straightforwardly for use in DST and PBA. These methods can be useful when the overall uncertainty about a variable is small.
3. Sensitivity analyses can also be applied within a DST or PBA analysis by hypothetically reducing epistemic uncertainty, aleatory uncertainty, or both. This approach can be used whatever the magnitude of the uncertainty about the input variables, but it requires decisions by the analyst on how the reduction should be defined and how the basic and residual uncertainty should be measured.

The hypothetical reduction of uncertainty may be done in different ways. In particular, the analysis might remove all uncertainty so that a parameter is held at some fixed value. Alternatively, the analysis might only pinch away the epistemic portion of the uncertainty and leave the parameter to vary according to some fixed probability distribution. Finally, it could exclude the aleatory uncertainty and consider only the consequences of epistemic portion of uncertainty about the parameter without any variability. The reduction of uncertainty resulting under these various hypotheses can also be measured in different ways. Although variance is often used as the measure of uncertainty in probabilistic analyses, it turns out that, when analyses are concerned with both types of uncertainty, variance is not the only, and may not be the best, possible measure of uncertainty. Other possible measures include the range, interquartile distance and related measures based on order statistics, and the area between the upper and lower bounds of an uncertain number.

Table of contents

1 Introduction 7

1.1 New methods for uncertainty analysis 9

2 The new methods *are* sensitivity analyses 14

3 Sensitivity analyses using uncertain numbers 16

4 Sensitivity analyses *within* the new methods 24

4.1 Effect of the arbitrariness in pinching 32

4.2 Different measures of uncertainty 34

5 Case study: dike revetment 37

5.1 Traditional probabilistic approach 39

5.2 What-if sensitivity analysis with DST and PBA 41

5.3 Second-order probabilistic analysis 44

5.4 Local derivatives 45

5.5 Reductions in uncertainty by pinching 47

6 The anova myth 49

7 Application to hard black box models 50

8 Conclusions and future directions 54

8.1 Future directions 56

9 Glossary 58

10 References 64

1 Introduction

Sensitivity analysis is a method to assess the sensitivity of a model to changes in its input parameters. If small changes in an input parameter result in relatively large changes in a model's output, the model is said to be sensitive to the parameter. The changes could be actual changes anticipated in the future, or potential changes that might occur, or merely hypothetical changes being considered as part of planning. Sensitivity analyses are usually conducted to understand how the conclusions and inferences drawn from a calculation or an assessment depend on its inputs, and also to focus future empirical studies so that effort might be expended to improve estimates of inputs that would lead to the most improvement in the estimates of the outputs. Many researchers find sensitivity measures useful as a further reality check on the plausibility of a model, or for addressing theoretical issues such as, for instance, whether the model's variables are related to each other in a reasonable way.

When applied to fixed models involving deterministic quantities, sensitivity analyses are often reported in terms of sensitivity indices that are related in some way to mathematical derivatives (tangents) of the response function with respect to an input of the function, or perhaps as graphs showing how the response function would vary as the value of an input parameter is altered. The derivative gives a local picture of how small changes in the input would be translated into consequences on the response. The graph can reveal a global picture of the effects of input changes of any magnitude. When the input parameters or the function itself are generalized from the deterministic case to probability distributions and stochastic functions, the questions of sensitivity analysis become entwined with related questions about how the actual variability in the input is propagated through the function and expressed as variation in its output. In deterministic analyses, the changes under discussion could be purely hypothetical or the result of engineering design or human intervention. But, in the probabilistic setting, the variation of the inputs might commonly be a consequence of natural processes, perhaps beyond human control.

Because of the obvious and fundamental importance of sensitivity analyses for the interpretation of uncertain calculations, there has been a confluence of attention to this issue from disparate analytical disciplines. Sensitivity analysis consequently has many manifestations in probabilistic analyses and there are many disparate approaches based on various measures of influence and response (Helton et al. 2006b; Cacuci 2003; Frey and Patil 2002; Frey 2002; Saltelli et al. 2000; Hamby 1994; Helton 1993; Iman and Helton 1988; and references therein). L'Ecuyer (1991) and Fu (1994) provided general reviews on the estimation of gradients in simulation studies. Cacuci et al. (2003; 2005) have promoted this approach, especially in nuclear engineering. Morgan and Henrion (1990) reviewed techniques of sensitivity analysis that have been applied to risk analysis models. One can crudely classify the techniques for sensitivity analysis into three categories: brute-force methods, regression methods, and calculus methods. By 'brute-force', we refer to an approach in which some quantity is repeatedly estimated for various values of parameters for which sensitivity estimates are sought. Both systematic and

random sampling can be applied to single-variable or mixed-variable designs. This approach works if the evaluation cost is moderate, but it often isn't for the probabilistic models encountered in complex analyses. The regression approaches include response surface methods (Myers 1971; Morton 1983; Downing et al. 1985; Kleijnen 1992; Myers 1999; Myers et al. 2004), the Fourier amplitude sensitivity test (Cukier et al. 1978), and logistic regression (Hosmer and Lemeshow 1989). The last of these can be used to directly estimate sensitivities of probabilities (McCarthy et al. 1995). Finally, the calculus approaches include manually engineered symbolic differentiation (Iman and Helton 1988), computer calculus (Oblow 1983a; 1983b), and automatic differentiation (Fischer 1993; Korivi et al. 1994; Griewank 1989; 2000). Only the first is immediately applicable to probabilistic models, but it can involve many person-months of effort (Iman and Helton 1988). General approaches for sensitivity analysis of discrete event dynamic systems are studied by Ho and Cao (1991), Glasserman (1991), Rubinstein and Shapiro (1993), and Pflug (1996), inter alia.

Uryasev (1994; 1995) described a general technique to compute sensitivities to infinitesimal changes in parameters for probabilities that are estimated with Monte Carlo methods. The approach is useful for a special class of functions—sums of expectations of indicator functions—of which probabilities are a special case (Uryasev 1997). The approach theoretically applies to arbitrary model parameters, notwithstanding mathematical complexity of the model, nonlinearities, or correlation structure among the random variables. Remarkably, the calculation requires no further iterations of the Monte Carlo simulation beyond those needed to estimate the probability itself, and can often be accomplished with negligible additional computation. Indeed, the sensitivities of any number of parameters can be computed simultaneously, representing a vast savings of computational effort compared with the traditional approach to Monte Carlo sensitivity analysis.

During the generalization from the deterministic to the probabilistic setting, sensitivity analysis seems to have come to mean different things to different analysts. For instance, many hold that sensitivity analysis is a study of how the variation in the output of a model can be *apportioned* to different sources of variation in the inputs (e.g., Saltelli 2003, Li et al. 2001; Rabitz and Alis 1999; Saltelli et al. 1999; Sobol' 1993; inter alia). This approach decomposes variance in the output in terms of the contributions from the several inputs. This is surely different from the earlier interpretation of sensitivity analysis as applied to deterministic models, in which there is no variation to apportion. The variance-decomposition approach is not the only possible extension of sensitivity analysis for the probabilistic setting.

Leamer (1990) defined (global) sensitivity analysis as a systematic study in which "a neighborhood of alternative assumptions is selected and the corresponding interval of inferences is identified". Many analysts consider such a study to be properly categorized under uncertainty analysis rather than as any part of sensitivity analysis, but Leamer is hardly alone in his usage. The phrase "Bayesian sensitivity analysis" is used by Bayesians to describe an analysis that, in fact, resolves the forward problem of uncertainty analysis and answers the question of how reliable inferences about model outputs are given the uncertainty about the inputs (Berger 1985; Lavine 1991). Andres

(2004) suggested that sensitivity analysis reveals what is causing the uncertainty of model outputs. Thus, sensitivity analysis seems very close to uncertainty analysis (which Andres distinguished as revealing how uncertain the model results are). Saltelli (2000) went so far as to characterize sensitivity analysis as the study of relationships between information flowing into and out from a model. Such an expansive characterization excludes hardly any aspect of quantitative modeling.

The literature on sensitivity analysis is huge and scattered throughout all the quantitative disciplines. Many of the methods in use are home grown within specialties and are not applications of a broader, external theory with a secure mathematical interpretation. The subject matter is experiencing a spurt of growth, or perhaps a metastasis, which is both exciting and daunting. There seems to be an explosion of different indices, measures and strategies for sensitivity analysis. Even the purposes to which sensitivity analysis is put are growing in number. The Wikipedia entry for sensitivity analysis lists six purposes, including determining not just which factors contribute most to the output variability and the region in the space of inputs where model variation is largest, but also the quality of the model definition and resemblance of the model with the process under study (http://en.wikipedia/wiki/Sensitivity_analysis). Several recent reviews of sensitivity analyses, notably including Saltelli et al. (2000), Saltelli et al. (2004), and Frey et al. (2003; 2004; Frey 2002) are virtually encyclopedia, both in length and number of subtopics. Even if the discussion is limited to the methods used for probabilistic calculations, there is a huge variety in the strategies and ideas. This report is not intended to be a comprehensive review of these disparate approaches. It offers, instead, a perhaps contrarian perspective on some of the issues that arise in sensitivity analyses under general uncertainty. In particular, it points out some of the wrinkles that arise when we apply sensitivity analysis to calculation problems with uncertain numbers which involve both epistemic and aleatory uncertainties.

The rest of this introduction section very briefly reviews the Dempster-Shafer evidence theory (DST) and probability bounds analysis (PBA). Section 2 argues that DST and PBA are themselves sensitivity analyses with considerable comprehensiveness and practical utility. Section 3 considers the use of standard methods based on derivatives for parametric sensitivity analysis in assessments involving uncertain numbers. Section 4 explores how a sensitivity analysis can be constructed within a DST or PBA assessment by hypothetically reducing uncertainty. Section 5 applies the ideas of the previous sections to a dike reliability assessment problem. Section 6 discusses the notion that variance is the only or best measure of uncertainty in non-deterministic sensitivity analyses. Section 7 describes the application of the methods to black-box models and section 8 draws some conclusions and points out some possible directions for future research.

1.1 New methods for uncertainty analysis

This section gives a synopsis of Dempster-Shafer theory (DST) and probability bounds analysis (PBA) and outlines the relationship between these new methods and interval analysis and probabilistic uncertainty analysis from which they are jointly derived. This synopsis is minimally sufficient to understand the rest of the report; see the

references below for a more discursive and thorough introduction to the use of these new methods in risk and uncertainty assessments.

There are two fundamentally different ways in which uncertainty about inputs is propagated through a mathematical model in order to estimate the reliability of the calculations or inferences derived from the model. One natural way is to bound the neighborhood of possible input values with interval ranges. Another natural way is to ascribe a probability distribution to the elements in this neighborhood of possible input values. In the context of a deterministic calculation, when the model involves uncertainty about the real-valued quantities used in the calculation, uncertainty analysis can be conducted via interval analysis (Young 1931; Dwyer 1951; Moore 1966; Alefield and Herzberger 1983; Neumaier 1990). Probability theory, implemented perhaps by Monte Carlo simulation, can also be used as an uncertainty analysis of a deterministic calculation because it yields a distribution describing the probability of alternative possible values about a point estimate (Iman and Helton 1988; Morgan and Henrion 1990; Helton and Davis 2000b; 2003). In the figure below these two possible paths are shown as right and left downward arrows respectively.

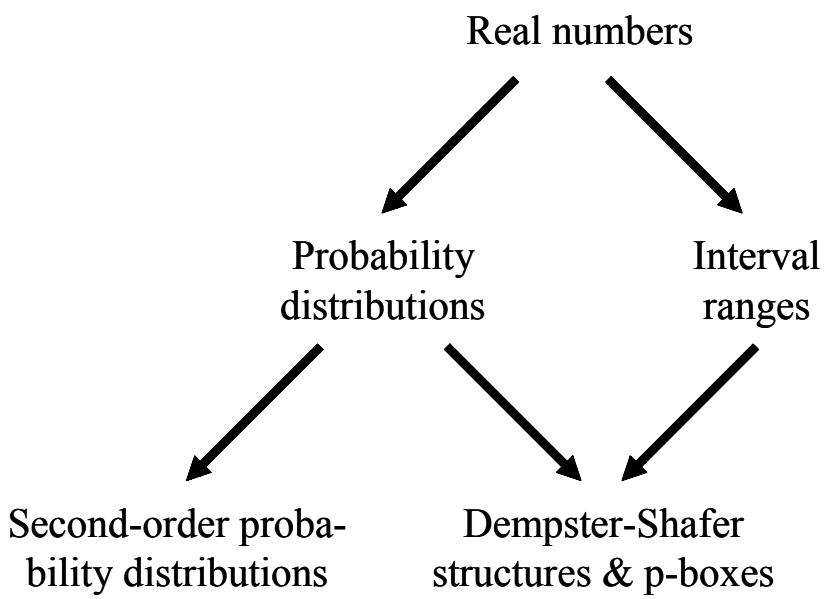

Figure 1. Relationships among kinds of uncertain numbers.

Of course, the calculations on which it might be desirable to conduct uncertainty analyses are not all deterministic. In fact, many of them are already probabilistic, as is the case in most modern risk analyses and safety assessments. One could construct a probabilistic uncertainty analysis of a probabilistic calculation. The resulting analysis would be a second-order probabilistic assessment (Hoffman and Hammonds 1994; Cullen and Frey 1999). However, such studies can be difficult to conduct because of the large

number of calculations that are required. It is also sometimes difficult to visualize the results in a way that is easily comprehensible. Alternatively, one could apply bounding arguments to the probabilistic calculation and arrive at interval versions of probability distributions. We call such calculations "probability bounds analysis" (PBA) (Ferson 1994; 2002; Ferson et al. 2003). This approach represents the uncertainty about a probability distribution by the set of cumulative distribution functions lying entirely within a pair of bounding distribution functions called a "probability box" or a "p-box". Mathematically, a p-box is the pair $[\overline{F}, \underline{F}]$, where $\overline{F}$, $\underline{F}$: $\Re \rightarrow [0,1]$, $\overline{F}(x) \leq \overline{F}(y)$ and $\underline{F}(x) \leq \underline{F}(y)$ whenever $x < y$, and $\overline{F}(x) \geq \underline{F}(x)$ for all x. A p-box can be identified with the set of distribution functions $F(x)$ such that $\underline{F}(x) \leq F(x) \leq \overline{F}(x)$ for all x values. PBA is an uncertainty analysis of a probabilistic calculation because it defines neighborhoods of probability distributions (i.e., the p-boxes) that represent the uncertainty about imperfectly known input distributions and projects this uncertainty through the model to identify a neighborhood of answers (also characterized by a p-box) in a way that guarantees the resulting bounds will entirely enclose the cumulative distribution function of the output. A probability distribution is to a p-box the way a real scalar number is to an interval. The bounding distributions of the p-box enclose all possible distributions in the same way that the endpoints of the interval circumscribe the possible real values.

Dempster-Shafer evidence theory (Oberkampf et al. 2005; Klir and Yuan 1995; Shafer 1976) has been widely studied in computer science and artificial intelligence, although it has never achieved wide acceptance among probabilists and traditional statisticians. In a discrete probability distribution on the real line, a nonzero probability mass is associated with each of the possible points of the distribution. All other values have a probability mass of zero and the probabilities for all the points in the discrete distribution add up to one. A Dempster-Shafer structure on the real line is similar to a discrete distribution except that the locations at which the probability mass resides are *sets of real values*, rather than precise points. These sets associated with nonzero mass are called focal elements, but unlike for discrete probability distributions, the focal elements can overlap one another. The correspondence between the probability masses and their associated focal elements is called the basic probability assignment. In this report, we focus on Dempster-Shafer structures whose focal elements are closed intervals. This restriction allows us to define a Dempster-Shafer structure on the real line as a collection of pairs each consisting of an interval and a mass $\{([x_1, y_1], m_1), ([x_2, y_2], m_2), \ldots, ([x_n, y_n], m_n)\}$, where $x_i \leq y_i$ for all i, $\Sigma m_i = 1$, and $y_i \neq y_j$ whenever $x_i = x_j$.

Yager (1986) considered bounds on the distribution function of a random real-valued quantity characterized by a finite Dempster-Shafer structure. For a finite Dempster-Shafer structure with basic probability assignment m and n focal elements $[x_i, y_i]$, i=1,…,n, with associated masses p_i, the upper bound for its distribution function is

$$\mathrm{CPF}(z) = \sum_{x_i \leq z} m_i \ .$$

The associated lower bound on the distribution function is

$$\mathrm{CBF}(z) = \sum_{y_i \le z} m_i \ .$$

Thus, the cumulative plausibility function is the sum of all masses associated with focal elements that overlap with or merely touch any value less than some value z, and the cumulative belief function is the sum of all masses associated with focal elements that are less than the value z. These functions bound a cumulative distribution function when there is insufficient information to specify it fully, and together they generalize the notion of a cumulative distribution function.

Yager (1986) also defined arithmetic operations between Dempster-Shafer structures that generalize the notion of convolution between distribution functions under the assumption of independence. The convolution is essentially a Cartesian product of the focal elements of the two operands. Given two Dempster-Shafer structures $\{([x_1, y_1], m_1), ([x_2, y_2], m_2), \ldots, ([x_n, y_n], m_n)\}$ and $\{([u_1, v_1], p_1), ([u_2, v_2], p_2), \ldots, ([u_k, v_k], p_k)\}$, the Cartesian product is the nk-element list $\{(f([x_1,y_1], [u_1,v_1]), m_1p_1), (f([x_1,y_1], [u_2,v_2]), m_1p_2), \ldots, (f([x_1,y_1], [u_k,v_k]), m_1p_k), \ldots, (f([x_2,y_2], [u_1,v_1]), m_2p_1), \ldots, (f([x_n,y_n], [u_k,v_k]), m_np_k)\}$, where f is the function that combines the two uncertain numbers now applied to their constituent focal elements. The masses are multiplied together under the independence assumption. Because $\sum_{j=1}^{k}\sum_{i=1}^{n} m_i p_j = 1$, this list is also a Dempster-Shafer structure, assuming any intervals in the product that happen to be identical coalesce into a single focal element and the associated masses are added. This structure is the uncertain number characterizing the generalized convolution between the two Dempster-Shafer structures.

The algorithm for convolving p-boxes under independence (Williamson and Downs 1990; Berleant 1993; 1996) is essentially identical to Yager's Cartesian product. The input p-boxes are discretized into Dempster-Shafer structures, the Cartesian product is computed, and the output p-box is reconstituted from the resulting Dempster-Shafer structure. Ferson et al. (2004) reviewed how to compute convolutions in both PBA and DST under other assumptions about dependence, and without any assumption about the dependence between the uncertain numbers.

PBA and DST are related to other forms of uncertainty analysis. They are both marriages of probability theory and interval analysis. As depicted in Figure 1, Dempster-Shafer structures and p-boxes can arise either by bounding probability distributions (the left path down) or by forming probability distributions of intervals (the right path). Yet these analyses are not simply an interval analysis with probability distributions. They are instead integrations of the two approaches that generalize and are faithful to both traditions. For instance, when PBA or DST is provided the same information as is used in a traditional probabilistic analysis (i.e., precise information about input distributions and their interdependencies), it will yield the same answers as the traditional analysis such as might be obtained by Monte Carlo simulation. When provided only range information about the inputs, they will yield the same answers as an interval analysis.

DST and PBA permit a comprehensive uncertainty analysis that is an alternative to second-order or nested Monte Carlo methods. PBA is very similar in spirit to Bayesian

sensitivity analysis (Berger 1985; Lavine 1991; which is also known as robust Bayes), although the former exclusively concerns arithmetic and convolutions, and the latter often addresses the issues of updating and aggregation. Unlike Bayesian sensitivity analysis, DST and PBA are always easy to deploy because they do not depend on the use of conjugate pairs to make calculations simple. Like Bayesian methods, however, DST and PBA may involve serious computational challenges, especially in large analyses. Rigorous bounds can be easy to compute, but getting those bounds optimally tight given available information can sometimes be difficult.

PBA and DST are practical approaches to computing with imprecise probabilities (Walley 1991). As in a Bayesian sensitivity analysis, imprecise probabilities are represented by a class of distribution functions. PBA is simpler than the general theory of imprecise probabilities because it defines the class solely by reference to two bounding distributions. It therefore cannot fully represent a situation in which there are intermediate distributions lying within the bounds that are excluded from the class. This means that p-boxes will often contain distributions that, if isolated and presented to an expert, would be rejected as quite far-fetched. However, in contexts of risk and safety assessments, this may not be a significant drawback if the analyst is principally concerned with the tail risks governing the probability of extreme events and not so much with the shapes of the distributions being enveloped. DST is generally more flexible than PBA in representing situations in which there are distributions that are unreasonable that are intermediate between plausible distributions, although much of this flexibility is lost if one restricts focal elements to be intervals.

Because DST and PBA marry probability theory and interval analysis, they treat aleatory uncertainty (variability) and epistemic uncertainty (incertitude) separately and propagate them differently so that each maintains its own character. In particular, aleatory uncertainty is propagated according to the rules of probability theory because it represents randomness, while epistemic uncertainty, on the other hand, is propagated according to the rules of interval analysis because it represents ignorance. The distinction between these two forms of uncertainty is considered very important in practical risk assessments (Helton 1994; Hoffman and Hammonds 1994; Paté-Cornell 1996; Helton 1997). PBA and DST are useful because they can account for the distinction when analysts think it is important, but the methods do not require the distinction in order to work. The two forms of uncertainty are like ice and snow in that they often seem to be very different, but, when studied closely, they can sometimes become harder and harder to distinguish from each other. For example, if an interval is combined in some arithmetic operation with a probability distribution, the epistemic uncertainty captured by the interval and the aleatory uncertainty captured by the probability distribution are folded together into the resulting p-box which, it can be argued, contains both epistemic and aleatory uncertainty in a way that can no longer be completely teased apart. An advantage of PBA and DST, and imprecise probability methods generally (Walley 1991), is that they can be developed in behavioral terms that do not depend on maintaining a strict distinction between the two forms of uncertainty which can be problematic.

PBA can be useful whenever the uncertainty about the marginal distributions can be characterized by interval bounds about their cumulative distribution functions. These

bounds can be specified using empirical information available about each distribution. For instance, if the parameters of a normal distribution can be given within interval ranges, best-possible bounds on the distribution are easy to construct. If the shape of the underlying distribution is not known, but some statistics such as the mean, mode, variance, etc. can be specified (or given as intervals), rigorous bounds can generally be constructed that are guaranteed to enclose the true distribution subject to the given constraints. Often these bounds will be optimally narrow given the stated information. In this case, the resulting p-boxes are distribution-free in the sense that they make no assumptions whatever about the distribution family (whether it is normal, lognormal, Weibull, etc.). Such bounds on distributions can then be combined according to the calculations in the assessment in arithmetic convolutions (addition, multiplication, minimum, etc.), magnitude comparisons (greater than, less than, inclusion), logical operations (conjunction, disjunction, etc.), and other mathematical transformations (logarithm, exponentiation, roots, etc.). The approach also allows the propagation of information or uncertainty about the dependencies among variables in a model that arise because of functional or statistical relationships among the variables (Ferson et al. 2004).

Although it is straightforward to ensure that bounds remain rigorous (sure to contain the true distributions) in sequential calculations, the 'best possible' nature of the bounds may be lost in some complicated calculations. Maintaining the optimality of the bounds is, in general, a computationally challenging task that can require other methods (Walley 1991). There can also be computational challenges in the application of DST and PBA to project uncertainty through black-box models (whose internal details are unknown and which can only be understood through sampling) and any model complicated by repeated uncertain variables (Manes 1982).

There have been several applications of the new methods to a wide variety of problems in risk analysis and uncertainty projection, including engineering applications (Oberkampf and Helton 2005), reliability studies for dikeworks (Hall and Lawry 2001), competing failure calculations for strong/weak link switches (Helton et al. 2004a; 2005b; 2006a); automotive design (Rekuc et al. 2006), aircraft reliability (Tonon et al. 1999); global warming (Kriegler and Held 2005), geological engineering (Tonon et al. 2000a,b), slope stability and landslide analysis (Rubio et al. 2004), human health risk assessments (MacDonald et al. 2002; EA 2002-2005), and endangered species viability analyses (Ferson and Burgman 1995).

2 The new methods *are* sensitivity analyses

Dempster-Shafer theory (DST) and probability bounds analysis (PBA) can be used to characterize the neighborhood of possible results that would be obtained from plausible alternative inputs in probabilistic calculations. This short section argues that, under a broad definition of sensitivity analysis, the new mathematical methods of DST and PBA used to calculate convolutions among uncertain numbers can be sensitivity analyses in their own right, even though they do not produce sensitivity rankings or coefficients. This section reviews how the new methods can be used to assess the quality of probabilistic models such as those developed in Monte Carlo simulations for risk analyses.

Sensitivity analysis is sometimes defined as a systematic study of how a model's output depends on its input. This definition is the sense implied when an analyst undertakes a "what-if" sensitivity study. Monte Carlo simulations and Bayesian analyses can be viewed as a kind of sensitivity analysis themselves (Helton and Davis 2002; Morgan and Henrion 1990; Iman and Helton 1985) in that they yield a distribution describing the variability about a deterministic point estimate. This section suggests that, likewise, DST and PBA are also kinds of sensitivity analyses when they are applied to deterministic calculations. In this argument, we are using the phrase "sensitivity analysis" as it has been used by several authors (e.g., "probabilistic sensitivity analysis" used by O'Hagan 2004; "global sensitivity analysis" used by Leamer 1990; "Bayesian sensitivity analysis" used by Berger 1985) to denote what-if studies that assess how a model's output depends on uncertainty about its inputs (rather than to quantify the importance of uncertainty sources on a model's output).

Many Monte Carlo simulations employ what-if sensitivity studies to explore the possible impact on the assessment results of varying the inputs. For instance, the effect of the truncating a infinite-tailed distribution at some finite upper value might be explored by re-running the model with various truncation settings, and observing the effect on the output statistic or distribution. The effect of particular parameter and probability distribution choices, and assumptions regarding dependencies between variables can also be examined in this way. Model uncertainty can be probed by running simulations using different models. However, such studies are often very difficult to conduct because of the large number of calculations that are required. While informative, this approach is rarely comprehensive because, when there are multiple uncertainties at issue (as there usually are), the shear factorial problem of computing all of the possible combinations becomes prohibitive. Usually, in practice, only a relatively tiny number of such analyses can be performed.

PBA and DST can be used to automate such what-if sensitivity studies and vastly increase their comprehensiveness (Ferson, 1994; 1996; 2001). They can produce rigorous bounds around the risk distribution from an assessment that enclose all the possible distributions that could actually arise given what is known and what is not known about the model and its inputs. For this reason, they can be used as a complementary quality assurance check on Monte Carlo simulation (Ferson, 1995; 1997). Because both methods are based on the idea of bounding rather than approximation, they provide an estimate of their own reliability (Berleant 1993; 1996; cf. Adams and Kulisch 1993). So long as the analyst can encode every step of a model calculation in terms of a sequence of transformations of and binary convolutions between the uncertain inputs, PBA and DST can comprehensively account for possible deviations in assessment results arising from uncertainty about

- distribution parameters,
- distribution shape or family,
- intervariable dependence, and even
- model structure.

Moreover, they can handle all of these kinds of uncertainties in single calculations that give a simple and rigorous characterization of how different the result could be given all

of the professed uncertainty. The requisite computations used in PBA and DST are actually quite simple and have been implemented in straightforward algorithms (Yager 1986; Williamson and Downs 1990; Berleant 1993; 1996; Ferson 2002; Ferson and Hajagos 2004). For a large class of problems of moderate size commonly encountered in risk analysis, the computations for DST and PBA can be faster than a numerically intensive sensitivity analysis with traditional methods (Iman et al. 1980; 1981a,b;; Iman and Conover 1982; Iman and Helton 1988; Morgan and Henrion 1990; Saltelli et al. 2000; Helton and Davis 2000b).

The advantages outlined above of the new methods are limited to models involving explicitly known calculations. PBA and DST cannot be rigorously applied to black box models, at least with algorithms currently available. Various sampling schemes have been proposed to extend DST and PBA to black boxes (e.g., Helton et al. 2004a,b; 2006c; Bruns et al. 2006), but, because they are necessarily approximation methods, they abandon the guarantee that the results will enclose the true distributions. This means that the sampling-based methods do not provide "automatic result verification" (sensu Adams and Kulisch 1993), although numerical simulations suggest that overall error rates can be made reasonably small if the black box permits many samples to be computed.

Furthermore, although it is often easy to apply the new methods to explicitly known models in a way that produces results that rigorously bound the possible output distributions (given the inputs), it can often be difficult to make the results *best possible* bounds whenever uncertain variables appear multiple times in a calculation as often occurs in complex models involving dependencies among subcomponents. In general, computing best possible bounds is computationally intensive, and optimality of the bounds becomes harder and harder to maintain as the size of the problem and the interconnectedness of its elements increase. In practical assessments, however, optimality may not be essential. For instance, if an assessment can show that the risks are surely below some threshold of concern, because the upper distributional bound of risk is below it, then no further study is necessary. Likewise, if the result indicates that the risks are surely intolerably high, because the lower distributional bound on risk is above the threshold, then the implied decision is also clear. In practice we find analyses producing results with clear implications for decisions and management to be surprisingly common, even in cases with large input uncertainties that might have been expected to cloud the results. This frees available analytical resources to be concentrated where they are needed most: on problems where the attendant uncertainty makes the decision unclear.

3 Sensitivity analyses using uncertain numbers

This section shows how some of the standard methods of differential calculus for evaluation of parametric sensitivity can be straightforwardly extended to work with uncertain numbers.

Several of the standard methods of sensitivity analysis employed for deterministic problems can also be used in the context of a probabilistic uncertainty assessment (Frey et al. 2003 Frey et al. 2004; EPA 2001, Section A). Although some of these methods do not have apparent analogs in DST or PBA (e.g., correlation analysis), many can be immediately generalized to work with Dempster-Shafer structures and p-boxes. For

instance, one of the most basic ways to evaluate sensitivity of an input variable is by computing the derivative of the output function with respect to it. For example, from the expression

$$x = \frac{abc + de}{f}$$

(which is similar in form to many of the simplest risk expressions encountered), the partial derivatives of x with respect to each variable are

$$\frac{\partial x}{\partial a} = \frac{bc}{f},$$
$$\frac{\partial x}{\partial b} = \frac{ac}{f},$$
$$\frac{\partial x}{\partial c} = \frac{ab}{f},$$
$$\frac{\partial x}{\partial d} = \frac{e}{f},$$
$$\frac{\partial x}{\partial e} = \frac{d}{f}, \text{ and}$$
$$\frac{\partial x}{\partial f} = \frac{-(abc + de)}{f^2}.$$

If the nominal values for these variables are the point estimates $a = 3$, $b = 4$, $c = 5$, $d = 8$, $e = 10$, and $f = 2$, one would estimate the sensitivity of x to changes in a to be $\partial x/\partial a = (4\times5)/2 = 10$. Using the respective formulas for all the variables and the same point values, one would compute the following sensitivities:

Variable	*Sensitivity*
a	10
b	7.5
c	6
d	5
e	4
f	−35

These partial derivatives are called sensitivity coefficients because they reflect how small changes in an input variable would precipitate changes in the output. The numerical results suggest that the quantity x is most sensitive to changes in the variable f, followed by variable a and then variable b. When the parameters have units, it is often desirable to express the sensitivities in a way that is insensitive to the units so that, for instance, changing from kilograms to grams doesn't increase sensitivity 1000-fold. It

might be reasonable to use the normalization $(\partial x/\partial z)(z/x)$ where z is one of the six input parameters. There is one important caveat about using partial derivatives as a measure of sensitivity: the results are local in nature. They represent the slope of how x changes in response to *infinitesimal* changes in an input parameter. If the relevant range for a parameter is broad, the sensitivity computed this way may begin to lose its meaning.

These calculations can be extended directly to the case in which the inputs are uncertain numbers. DST or PBA can be applied directly to the formulas for the partial derivatives to obtain Dempster-Shafer structures or p-boxes estimating the sensitivities. Suppose that the nominal values for the variables are given as

```
a = [2.5, 3.5],
b = normal(mean=4,stdev=0.02),
c = p-box(min=4.3, max=5.2, mean=5),
d = 8,
e = p-box(mean=10, variance=0.001), and
f = uniform(min=1.9, max=2.2),
```

which include two precise probability distributions, two distribution-free p-boxes, one interval and one point value. These uncertain numbers are depicted in Figure 2. (The quantity *d* is a degenerate uncertain number because it lacks both epistemic and aleatory uncertainty.)

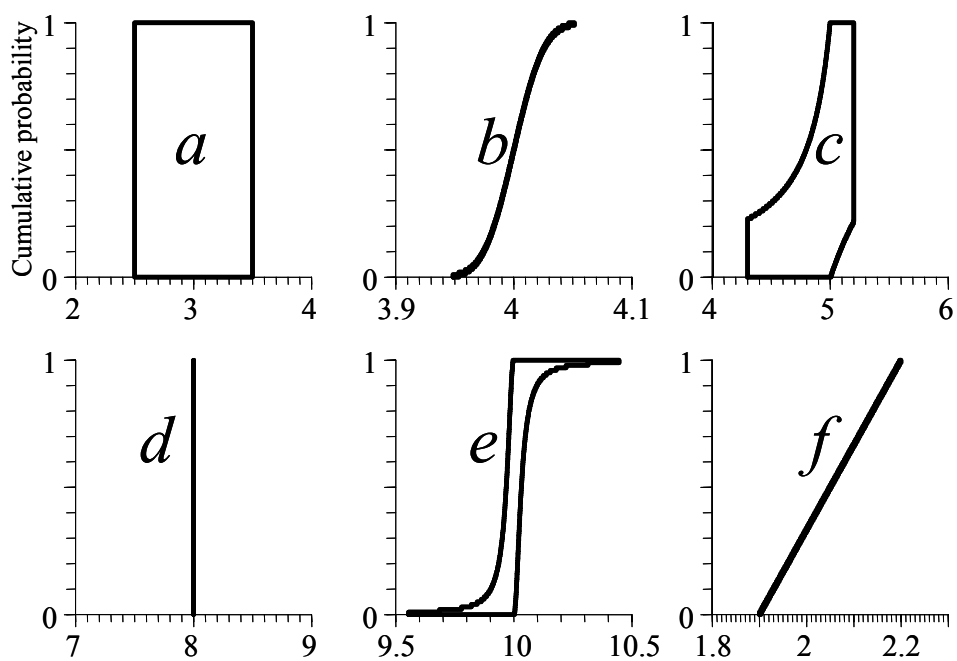

Figure 2. Six uncertain inputs.

If these inputs are propagated through the six formulas for partial derivatives of the input variables, one obtains an uncertain number for each variable, representing an estimate of the sensitivity of x to changes in that variable. The sensitivities were computed semi-analytically, without any Monte Carlo sampling, using the RAMAS Risk Calc software (Ferson 2002) and specially prepared scripts for the R environment (R Development Core Team 2004). The results are displayed in Figure 3. Because these estimates contain both epistemic and aleatory uncertainty, they cannot be fully characterized by scalar numbers. Even their summary statistics such as the mean and variance are usually uncertain. These statistics are intervals rather than scalar values because the sensitivity estimates are general uncertain numbers, except for $\partial x/\partial e$ which is a probability distribution (because it is a function of only d and f). The ranges, means, medians and variances for the uncertain numbers depicted in Figure 3 are given in the table below with two significant digits and outward-directed rounding.

Sensitivities

Coefficient	*Range*	*Mean*	*Median*	*Variance*
$\partial x/\partial a$	[7.7, 12]	[9.7, 9.8]	[9.1, 11]	[0.16, 0.72]
$\partial x/\partial b$	[4.8, 9.6]	[6.1, 8.6]	[5.7, 8.9]	[0.064, 2.1]
$\partial x/\partial c$	[4.4, 7.5]	[4.8, 6.9]	[4.8, 6.9]	[0.042, 1.1]
$\partial x/\partial d$	[4.3, 5.5]	[4.8, 4. 9]	[4.8, 5.0]	[0.041, 0.045]
$\partial x/\partial e$	[3.6, 4.3]	[3.9, 3.9]	[3.8, 4.0]	[0.026, 0.029]
$\partial x/\partial f$	[−43, −25]	[−36, −31]	[−36, −30]	[6.7, 17]

These results also reveal the relative importance of f compared to the other variables. Note, however, that it will no longer generally be possible to strictly rank the sensitivities. Because they can overlap one another, it is not possible to define an ordering* for uncertain numbers. For instance, in the point sensitivity analysis, the quantity seemed to be more sensitive to c than to d. When epistemic and aleatory uncertainty are taken into account, this difference is not so clear because the sensitivities overlap.

*An ordering could, however, be defined with reference to some scalar characteristic of the uncertain numbers such as the midpoint, upper bound on the 95th percentile, largest possible value, etc.

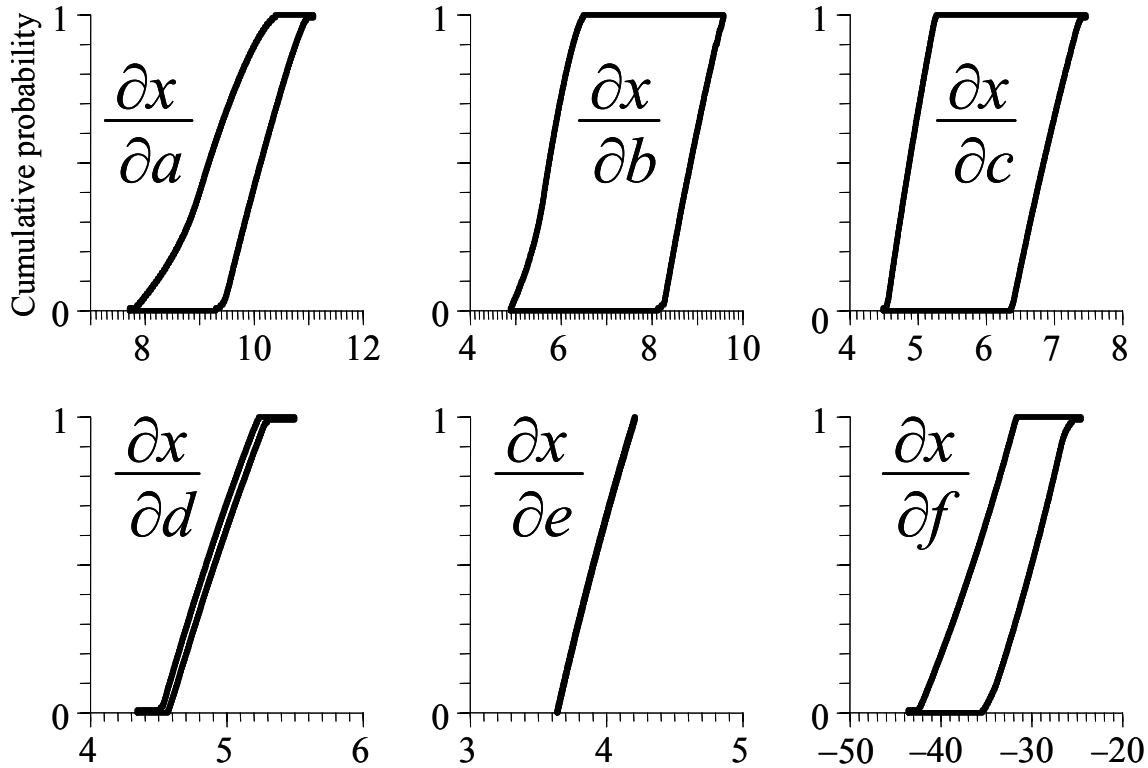

Figure 3. Sensitivities of the function to the six uncertain inputs.

Figure 4 offers an alternative depiction of the uncertainty about how the function x changes as each of the input variables varies. In each of the six graphs the value of x is plotted as a function of an input variable varying over its possible range. In each graph, there is a series of five curves illustrating the relationship between x and the input. Each such curve was created by holding the other inputs fixed and varying only the variable named. However, because there is uncertainty about what the other variables actually are, we have randomly selected five sets of inputs, each from its possible range, and plotted the collection of five curves. (The random selections were independent from graph to graph.) The resulting six graphs show, therefore, how the function is influenced by each of the six variables separately, but in the context of substantial uncertainty from the other variables. The plot for variable d is just a scatter of points because d is a constant and ostensibly has no uncertainty. Notice the slight curvature of the lines, especially pronounced in the plot for f. The curves within each graph are not merely shift translations of each other; the lines are not all parallel. The ordinates are all on the same scale, so one can see right away that varying the input variable b does not promise to give one much control over the output x. The variable a, on the other hand, seems to offer greater control over x with minimal confounding uncertainty from the other variables, although an analyst might want to see many more than five replicate plots before drawing this conclusion.

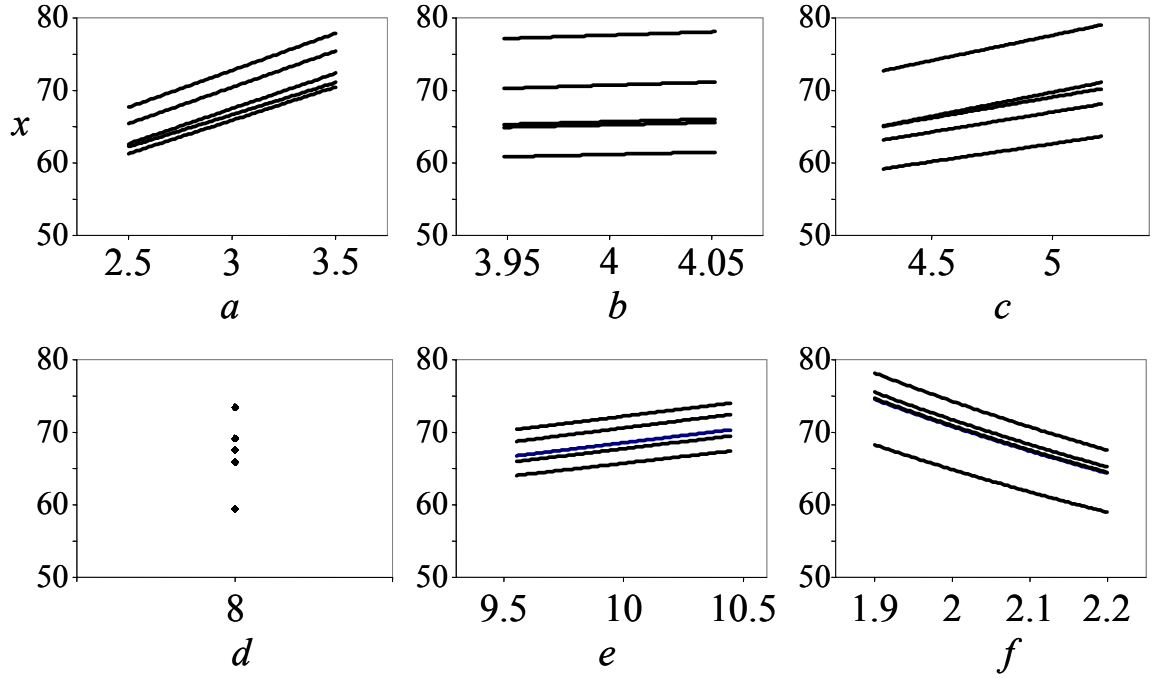

Figure 4. Variation of *x* as a function of its inputs.

It would also be possible, albeit even more cumbersome, to depict the total uncertainty about *x* in a comprehensive way as the input variables are varied. The ladder plot in Figure 5 shows how the total uncertainty about *x* varies as the input variable *a* is set to a constant, for five different values of this constant over its possible range [2.5, 3.5]. The output uncertain numbers are simple shift or multiplicative translations of each other. The uncertain numbers can often overlap confusingly if they were plotted on a single graph, but it might be practical to create animated graphs that would be useful to analysts with modern visualization software.

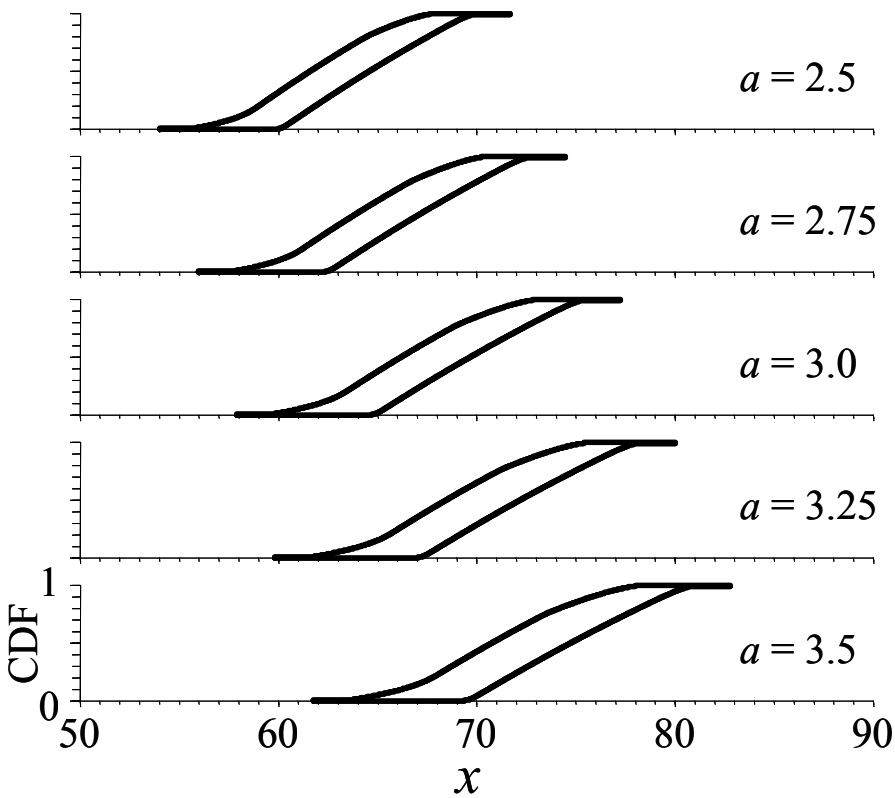

Figure 5. Variation in uncertainty about *x* as *a* is varied over its possible range.

The calculations in the above example assumed mutual independence among all the inputs, but this assumption is not essential and could be relaxed to reflect empirical evidence and the analyst's uncertainty about the dependencies.

We have now considered three different characterizations of sensitivities for functions involving uncertain numbers. Figure 3 shows local derivatives, which refer to how *x* changes in response to *infinitesimally small* changes in a variable. In contrast, the curves depicted in Figure 4 show these changes over wide ranges of the input variables. Figure 5 shows us what conclusions we can draw about the output in the face of the uncertainties of the inputs if we could somehow perfectly control one of them. These figures are telling us slightly different things. For example, the results in Figure 3 clearly indicate the function is most sensitive to variable *f*, yet, as we've already mentioned, Figure 4 suggests that controlling variable *a* might be preferable as a management strategy if doing so could be cost-effective.

The caveat about computing sensitivities as partial derivatives also applies when they are used with uncertain numbers: the results reflect sensitivities to local changes. If the range of an uncertain number is broad, the estimates may be hard to interpret. This issue is also present but cryptic in deterministic sensitivity analyses. How much uncertainty can be present before the results of the analysis become unreasonable depends on the function being analyzed and its inputs. One benefit of explicitly accounting for

uncertainty is that the true imprecision of the resulting sensitivities and their ranks becomes self-evident. It is possible, for instance, that the uncertain number obtained by computing the sensitivity coefficient with DST or PBA could straddle zero. This would mean that the uncertainty surrounding the analysis prevents us from being sure whether the slope of the tangent line is positive or negative.

What would it mean if the uncertain number characterizing the sensitivity coefficient computed as a local derivative were to straddle zero? At its simplest, the uncertain number represents the set of slopes of the lines that are tangent to the output or response function at different values of the input over some range of possibilities. This is illustrated in Figure 6 in which there are two examples. The upper, left plot is a cartoon of a simple univariate function (shown as a gray curve) and a range of uncertainty for its input (shown as a black horizontal line segment). A similar cartoon for a more complicated function with a range of uncertainty for its input is depicted in the upper, right plot. The range of the input for the left function is narrower than for the right, and the left function is closer to linear over its input range than is the right function. (Assuming all functions we will deal with are analytic, then Taylor's theorem suggests that the narrower the input range, the closer to linear the function will appear.) Below each of these two plots are the same plots drawn together with tangent lines for the function at several points over the respective ranges of the inputs. For the function on the left, these slopes are a tidy collection of whiskers, but for the function on the right, the slopes are an unruly jumble including both positive and negative slopes having variously small and large magnitudes. The uncertain numbers computed as sensitivity coefficients using the methods of PBA and DST characterize all of the slopes in such collections of slopes, whether they are tidy or not. For this reason, these uncertain numbers characterize how nonlinear a function is over the range considered. If the sensitivity coefficient is a narrow uncertain number with little uncertainty, then the underlying function must be close to linear over this range. If, however, the uncertainty is quite wide, then the function could be strongly nonlinear in the sense that it exhibits broad changes in the slope of its tangent lines. If the wide uncertain number was computed to be best possible, then one can conclude that the function *is* strongly nonlinear, and not merely possibly so.

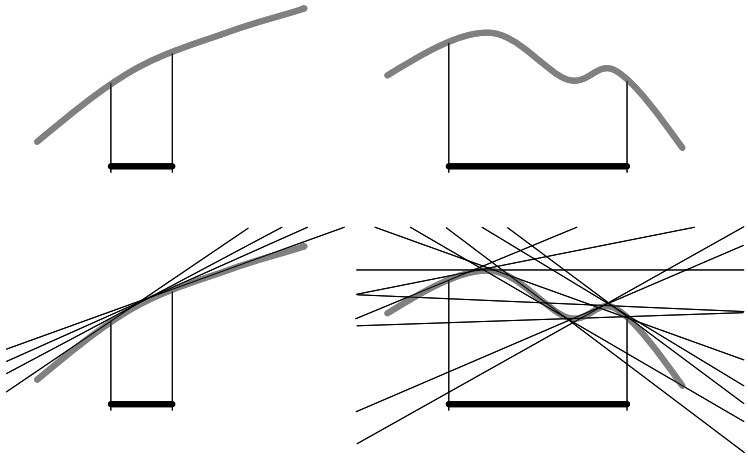

Figure 6. Pencils of tangent lines to simple (left) and complex (right) functions.

4 Sensitivity analyses *within* the new methods

It is possible and often of interest to perform a sensitivity analysis on the results of an assessment conducted within DST or PBA. This section explores the use of studies that assess the hypothetical impact on result uncertainty of additional empirical knowledge.

One of the fundamental purposes of sensitivity studies is to learn where focusing future empirical efforts would be most productive. This purpose requires estimating the value of additional empirical information (Morgan and Henrion 1990). Of course, the value of information not yet observed cannot be measured, but it can perhaps be predicted. One strategy to this end is to assess how much less uncertainty the calculations would have if extra knowledge about an input were available. This might be done by comparing the uncertainty before and after "pinching" an input, i.e., replacing it with a value without uncertainty. Of course, one does not generally know the correct value with certainty, so this replacement must be conjectural in nature. To pinch a parameter means to hypothetically reduce its uncertainty for the purpose of the thought-experiment. The experiment asks what would happen if there were less uncertainty about this number. Quantifying this effect assesses the contribution by the input to the overall uncertainty in a calculation.

The estimate of the value of information for a parameter will depend on how much uncertainty is present in the parameter, and how it affects the uncertainty in the final result. The sensitivity could be computed with an expression like

$$100\left(1-\frac{\mathrm{unc}(R)}{\mathrm{unc}(T)}\right)\%$$

where T is the baseline value of the expression, R is the value of the expression computed with an input pinched, and unc( ) is a measure of the uncertainty of the answer. The result is an estimate of the value of additional empirical information about the input in terms of the percent reduction in uncertainty that might be achieved in the expression when the input parameter is replaced by a better estimate obtained from future empirical study. The pinching can be applied to each input quantity in turn and the results used to rank the inputs in terms of their sensitivities. In principle, one could also pinch multiple inputs simultaneously to study interactions.

The unc( ) measure is analogous to variance as it is used in variance-based methods of global sensitivity analysis (Saltelli et al. 2000). There are many possible ways to define unc( ) to measure uncertainty. In the context of DST or PBA, one obvious measure is the area between upper and lower bounds of the p-box, which is equivalent to the integral of the difference between the cumulative plausibility function and the cumulative belief function for a Dempster-Shafer structure. As the uncertain number approaches a precise probability distribution where all epistemic uncertainty has evaporated and only the natural variability remains, this area approaches zero. An analyst might also elect to define unc( ) as variance or some other measure of dispersion, or perhaps the heaviness of the tails (in the sense of Hettmansperger and Keenan 1980) of

the uncertain number. Using different measures will obviously allow the analyst to address different questions in a sensitivity analysis. If the measure of uncertainty is a scalar quantity (i.e., a real number), then the sensitivities that come from the analysis will also be scalars and can be ordered.

It is worth noting that, unlike the factorizations used by variance-based sensitivity analyses (Saltelli et al. 2000), these reductions will *not* generally add up to 100% if the reductions for all of the input variables are summed. This is the general situation, and variance-based partitions of uncertainty are unusual among possible measures of uncertainty. This issue is more fully discussed in section 6.

There are also multiple possible ways to pinch uncertainty. Pinching in different ways can result in strongly different estimates of the overall value of information. Several strategies are possible in estimating sensitivities from comparative uncertainty assessments:

(i) replace an input with a point value,
(ii) replace an input with a precise distribution function,
(iii) replace an input with a zero-variance interval, or
(iv) replace an input with an uncertain number with smaller uncertainty.

The first three strategies are extreme cases of the fourth. Replacing an uncertain number with a precise probability distribution would be pinching away the epistemic uncertainty about the distribution. Replacing an uncertain number with a point value would be pinching away both the epistemic uncertainty and the aleatory uncertainty of the quantity. For inputs that are known to be variable (variance greater than zero), such a pinching is counterfactual, but it may nevertheless be informative. In particular, it may be especially useful in planning remediation strategies or manufacturing designs.

The third strategy merits some contemplation. Normally, the estimate of variance associated with an interval $[a,b]$ is the interval $[0, (b-a)^2/4]$. This is to say that the variance of any distribution whose support surely lies with the range $[a,b]$ must have a variance no larger than a quarter of that range squared, which would be the case if the distribution had half its mass at a and the other half at b. Its lower bound is zero because the true distribution might be any of many Dirac delta distributions (essentially, constants) within the range. In some situations, it may be reasonable to replace an uncertain number with another uncertain number similar in shape to an interval but prescribed to have a variance of zero. Such a pinching is hypothesizing that the quantity had a Dirac delta distribution whose value is within the interval. The effect of this would be to pinch away the aleatory uncertainty but leave the epistemic uncertainty. Such a replacement might be reasonable for uncertain numbers having a core.*

This approach of pinching inputs and recalculating the assessment is familiar to Monte Carlo analysts (e.g., Iman 1987; Iman and Hora 1990; Helton et al. 1995).

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*A p-box's core, if it exists, is the region along the abscissa for which the upper bound of the p-box is one and lower bound is zero. The core of a Dempster-Shafer structure is the intersection of all the focal elements. The core of an interval is the interval itself.

Although they know that it has pitfalls*, many analysts routinely conduct sensitivity studies of the proportional contribution of variability in each variable to the overall variability in the calculated risk distribution (e.g., Helton et al. 1995; 1997a,b). To determine the effects of variability in a Monte Carlo simulation using this method, each variable containing variability (i.e., expressed as a probability distribution) is reduced in turn to its mean or other appropriate point value, and the simulation is repeated. The measure of sensitivity is often the proportional effect of variability in each variable on the model, which is computed as a function of the variance in the output distribution from each of the simulations divided by the variance in the output distribution from the baseline model result. Although the general idea of pinching is known to Monte Carlo analysts, the notions of pinching to a precise distribution and pinching to a zero-variance interval have no analog in Monte Carlo sensitivity analyses.

The rest of this section of the report will describe eight case studies that explore the effect of these kinds of pinching on some simple synthetic numerical examples. Figure 7 shows the first two case studies which illustrate pinching to precise distributions. The top panel of the figure depicts the addition of two uncertain numbers *A* and *B* (assuming independence). This is the baseline case against which the pinchings will be compared. Uncertain number *A* is specified as a uniform distribution whose minimum is somewhere between 4 and 5, and whose maximum is between 5 and 6. Uncertain number *B* is specified as a normal distribution with unit variance whose mean is a value between 8 and 9. (Its tails were arbitrarily truncated at 5.4 and 11.6.) The uncertain number on the right labeled *A*+*B* is the envelope of all sum distributions that are the convolutions of a distribution drawn from uncertain number *A* and a distribution drawn from uncertain number *B*. The area between the upper and lower bounds for the sum *A*+*B* is 2.12. The middle panel of the figure shows the first pinching. Notice that it is labeled as the first of our eight little case studies. The uncertain number *A* is pinched to the precise uniform probability distribution between 4.5 and 5.6, which lies entirely within the uncertain number depicted above it. Under this pinching, the addition with *B* (which is still the same uncertain number) yields the resulting uncertain number shown at the far right on the middle panel. This uncertain number has an area of about 1.12. The percentage reduction in this area compared to that of the uncertain number for the sum shown on the top panel is about 47%. This percent, which labels the sum on the middle panel, represents the sensitivity measure for pinching the variable *A* to a precise probability distribution. The bottom panel of Figure 7, labeled as the second study, shows the reduction of uncertainty (area) for the sum *A*+*B* from pinching the uncertain number for *B* to a precise distribution instead. In this case, *B* is pinched to a normal distribution with a mean of 8.5 and unit variance. Compared to the baseline case in the top panel, the area is also reduced by around 47%. Thus, the reductions in uncertainty from pinching either *A* or *B* are essentially the same in this example. This need not always be the case.

Given the differences between the inputs *A* and *B*, one might in fact think that the reductions in uncertainty from pinching them are surprisingly similar. The variance of *B*

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*These pitfalls include arbitrariness of the pinching target, complications of handling pairwise and higher-order sensitivities, and susceptibility to gaming when multiple interests or analysts are involved.

is unity by construction, and pinching does not change the variance. The variance of *A* is 0.0833, which is much smaller than that of *B*. Even if we had pinched to some other uniform distribution consistent with *A*, its variance could only be as large as 0.333 (which would be obtained if *A* were pinched to the uniform distribution over [4,6]). Although the operation is addition, consideration of the variances or even the change in variance does not predict the reductions in uncertainty that were measured as the area between the bounds on the output. This example shows that variance is not a good measure of the importance of the epistemic uncertainty associated with an input, at least relative to how much its elimination might reduce the epistemic uncertainty of the result as measured by the breadth of its bounds. Another important dissimilarity with variance is that the percent reductions from *A* and *B* do not add up to 100%. Although the reductions are measures of the importance of additional information for each of the inputs, they are not a partition constrained to sum to unity, even for additive models under independence. These observations make it clear that the use of breadth between the upper and lower bounds is not at all similar to the analogous use of variance in probabilistic sensitivity analyses.

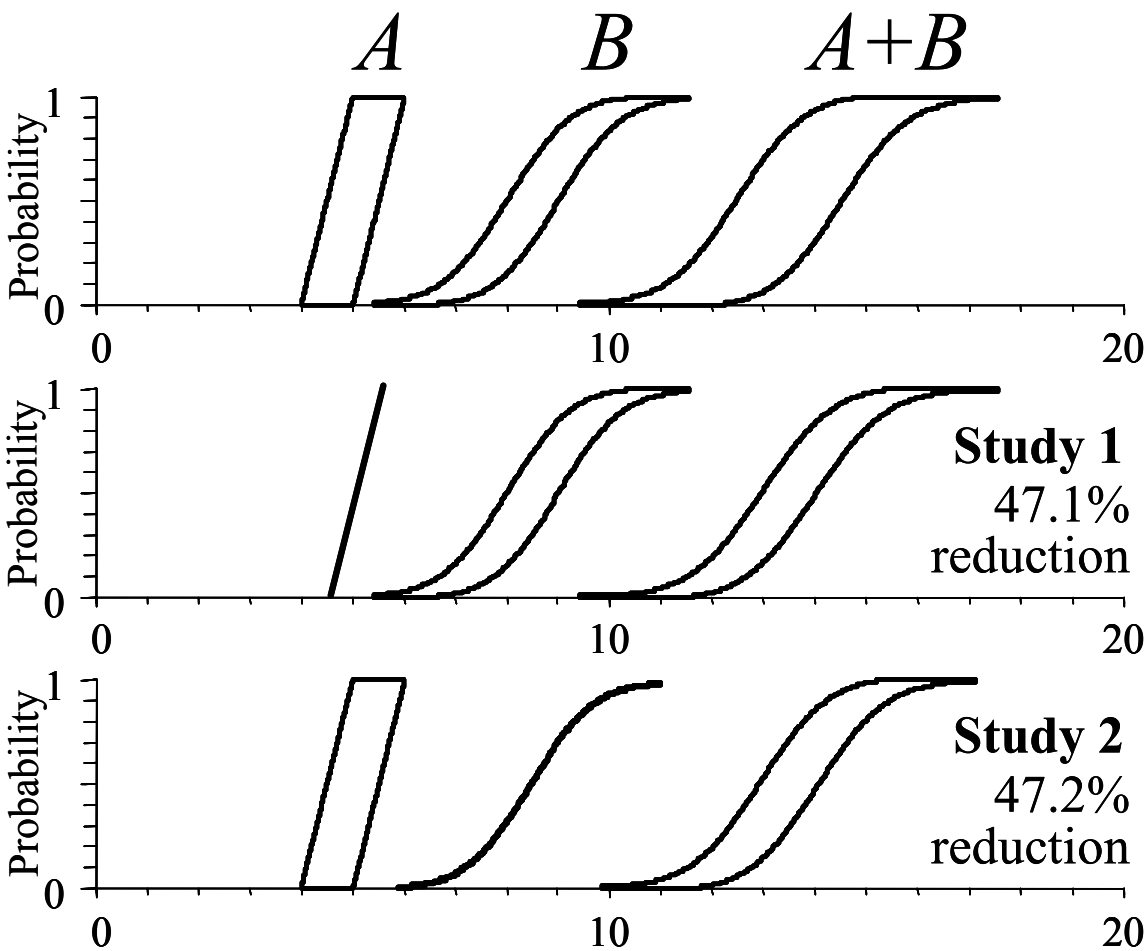

Figure 7. Sensitivity analyses by pinching uncertain numbers to distributions.

Figure 8 shows three more studies, which are a similar set of sensitivity analyses based on pinching epistemic uncertainty. The calculation for the baseline case in this figure (shown in the top panel) was made without making any assumption about the dependence between the variables *A* and *B*. For this reason, even though the uncertain numbers for the variables *A* and *B* are just the same as were used in Figure 7, the area of the sum grows to about 3.05. This result is called the Fréchet case or the "dependency bounds" (Williamson and Downs 1990) for the sum. It represents ignorance not only about the input distributions but also about their intervariable dependence (see Ferson et al. 2004 for a discussion of the Fréchet case and calculations under different assumptions about dependence). The second panel of Figure 8 depicts pinching the uncertain number for the variable *A* to a precise distribution and its consequence for the resulting uncertainty about the sum. The third panel likewise shows the pinching for variable *B*. Both panels are annotated with the percent reduction in the area of the uncertain number for the sum compared to the baseline case in the top panel. The bottom panel shows the effect of pinching the dependence from the Fréchet case of assuming nothing about dependence to assuming independence. The dependency bounds of the baseline case are replace with the bounds that would be obtained by assuming *A* and *B* are independent. (The pinching could have specified any particular dependence.) These studies illustrate that the pinching can be carried out with dependency bounds that do not make assumptions about intervariable dependence, and even more interestingly, that ignorance about dependence can itself also be pinched. Such pinching is a special case of strategy (iv) mentioned on page 25 because the uncertain joint distribution for *A* and *B* is replaced by a precise joint distribution that specifies their dependence.

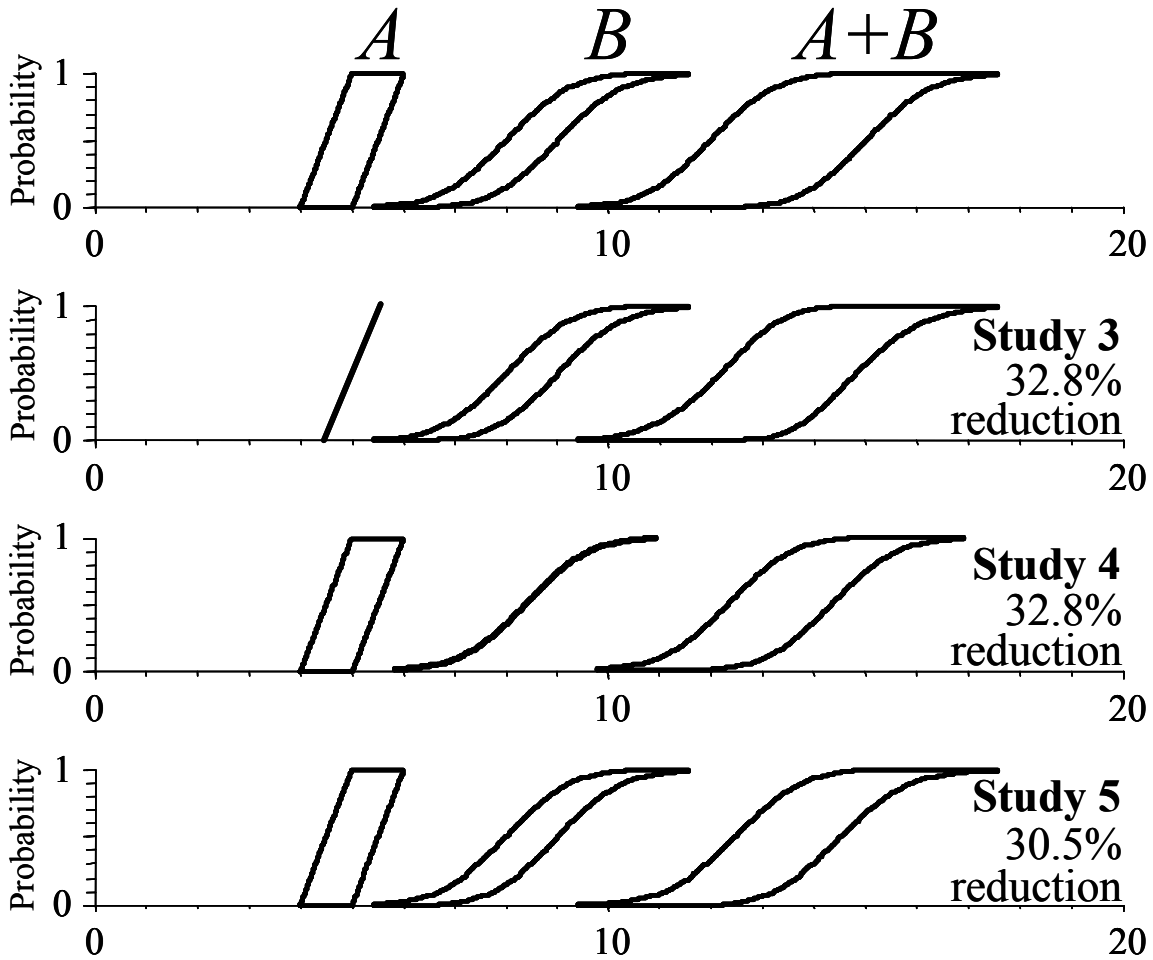

Figure 8. Sensitivity studies without dependence assumptions.

Figure 9 shows two more hypothetical sensitivity studies. The baseline case in the top panel is identical to the baseline case shown in Figure 7, but in this study, the uncertain numbers are pinched to scalar values. The second and third panels of Figure 9 depict the additions resulting from pinching one of the addends to a point value. The observed percentage reduction in the area of each resulting sum compared to the baseline case is shown beside its uncertain number. The top panel in Figure 9 depicts a baseline case that assumes independence between *A* and *B*. What would the reductions in uncertainty have been if the baseline calculation had not assumed independence? The pinchings would have yielded exactly the same results, simply because dependence assumptions have no effect when either of the addends is a point. Thus, the lower two panels of Figure 9 would look exactly the same, but the percent reductions in area would have been different. The reason for this is that, if the baseline calculation had not assumed independence, then the baseline uncertainty about the sum *A*+*B* would have been greater (area = 3.05, compared to 2.12 under independence). That would make the rightmost uncertain number in the top panel of Figure 9 noticeably wider. Therefore the reductions in uncertainty by pinching to a point would have been somewhat greater than

they were for the independent case. Instead of 50.4% and 52.36% reductions, pinching the variables *A* and *B* to points under no assumption about dependence would have respectively yielded 65.54% and 66.9% reductions in uncertainty as measured by the area within the resulting uncertain numbers.

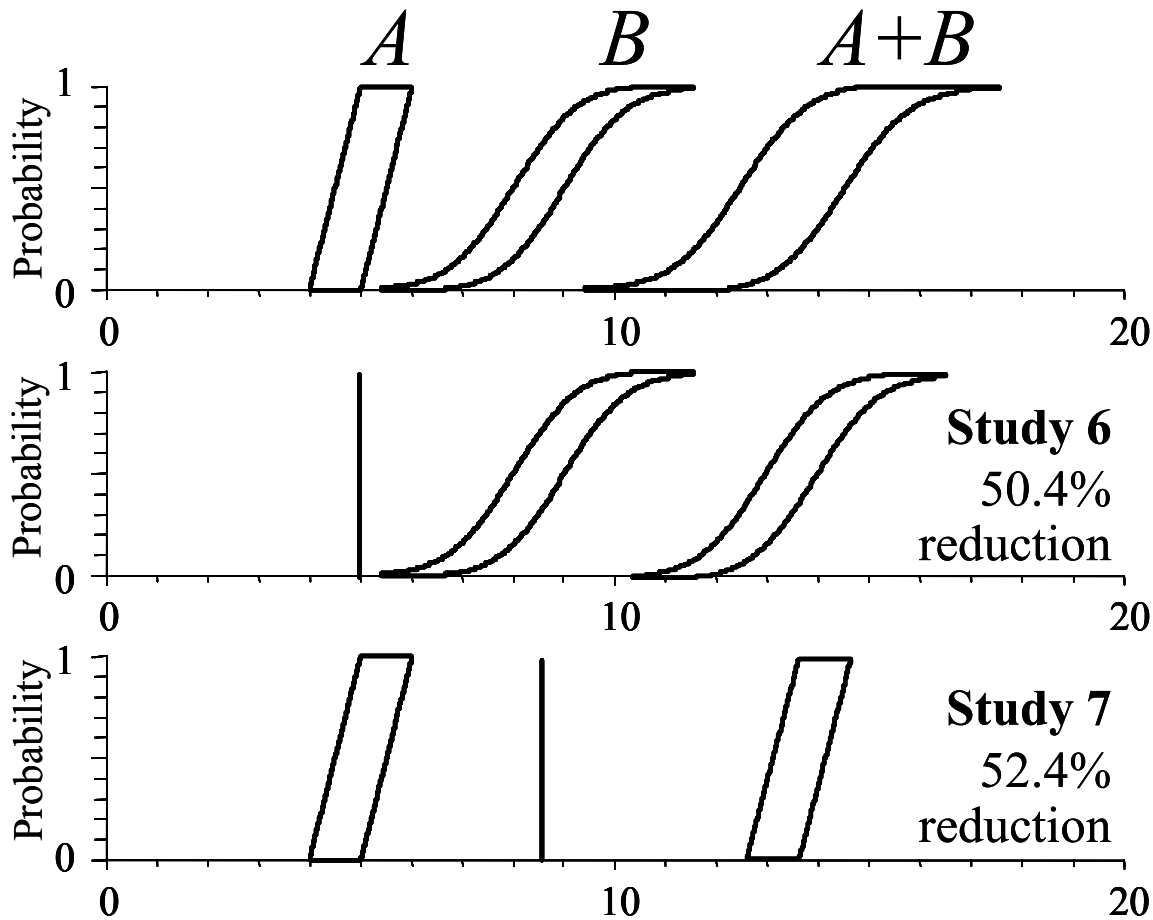

Figure 9. Pinching an uncertain number to a point value.

Figure 10 shows a final example sensitivity study. The baseline case in the top panel consists of an uncertain number for a quantity *A* about which the minimum is known to be 4, the maximum is known to be 7, and the mean is only known to be between 5 and 6. (When such information is obtained from published papers, it is easy to imagine that no other reliable information might be available about this variable.) The uncertain number labeled *A* is the best possible envelope on all distributions having this range and mean (Ferson et al. 2002). The uncertain number labeled *B* is the best possible envelope on the set of triangular distributions whose minimum is between 7 and 9, whose maximum is between 9 and 12, and whose mode is between 8 and 10. (These values were chosen to simplify the plots for these examples; in practice, the uncertain numbers could have arbitrary constraints.) The uncertain number for the distribution of the sum *A*+*B* from

these two inputs is shown at the far right of the top panel. This calculation assumed the variables were independent.

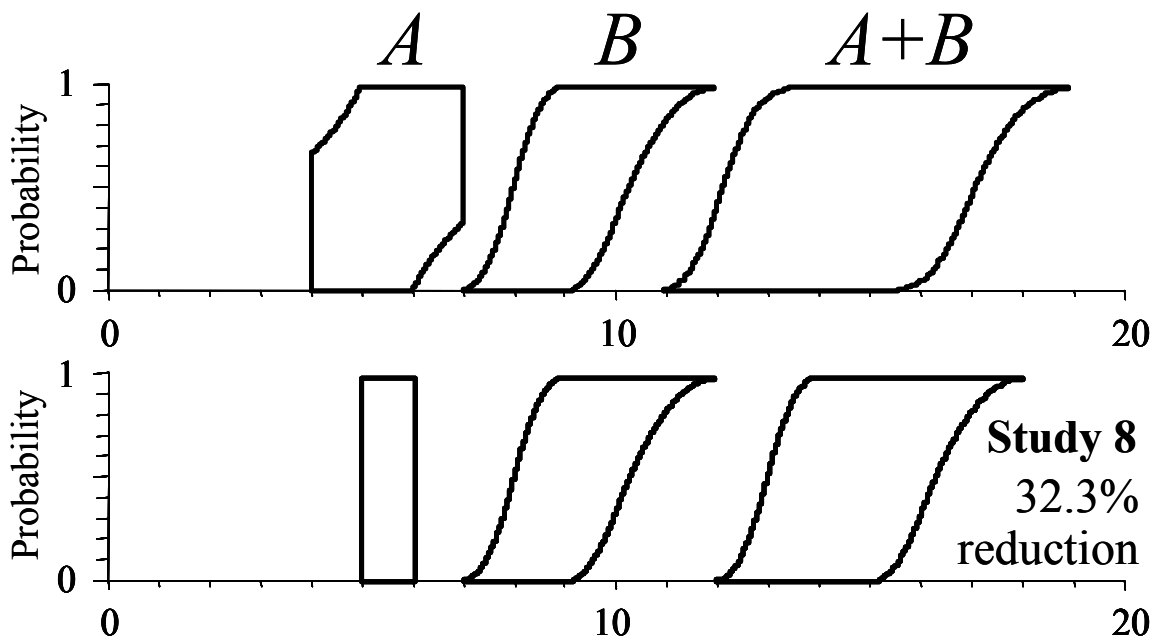

Figure 10. Pinching an uncertain number to a zero-variance interval.

In study 8, depicted in the lower panel of Figure 10, the uncertain number for the variable *A* was pinched so that it has zero variance. This removes the aleatory uncertainty from *A* but not the epistemic uncertainty about this quantity. Notice that in the lower panel the uncertain number for *A* has contracted to an interval. The reason is that a distribution for a quantity that has no aleatory uncertainty must have a degenerate Dirac delta distribution that is simply a spike. The only such distributions that fit within the prescribed uncertain number for *A* are those within its core between 5 and 6. This core therefore becomes the pinched representation of the quantity without variability. In addition to the visible contraction of the uncertain number, its variance, which had been a value known only to be between 0 and 2.25, was also set to zero. When the uncertain numbers are combined in study 8, they produce the uncertain number shown on the far right of the lower panel. The percentage reduction in the area of the resulting sum compared to the baseline case is about 32%. This is the reduction in the overall uncertainty as measured by the breadth of the uncertain number, but there are other ways to measure uncertainty, such as variance for instance. In the baseline case, the possible variance of the distribution of the sum *A*+*B* could be any value between 0 and 3.42. In the pinched case, this variance must be lower than 1.2. Thus the variance of *A*+*B* could be reduced by nearly 66% by removing the variability in the quantity *A*.

Pinching to remove variability such as we have described here can be applied to any uncertain number that has a core, that is, to any uncertain number for which the left

(lower) bound on the 100th percentile is smaller than the right (upper) bound on the zeroth percentile. That is, this kind of pinching is possible if a constant spike will *fit* inside the p-box or, equivalently, if the intersection of all the focal elements of a Dempster-Shafer structure is not empty. Uncertain numbers without a core represent quantities that, apparently, must vary and therefore could not be any single scalar value. It is clear that this kind of pinching is quite different from the pinching exemplified in Figure 9, where a distribution could always be replaced by a fixed, scalar value. The two approaches are based on completely different conceptions of constancy. The present approach asks what consequence there would be for our conclusions if a quantity which we had thought might be varying were discovered to actually be fixed. The previous kind of pinching asks about the effect of *holding* a varying quantity to some fixed value. Presumably, the process underlying this approach could only be actualized by some engineering intervention designed to regulate the quantity's magnitude. These are obviously different kinds of questions for a sensitivity analysis, and both seem to be useful for different purposes.

4.1 Effect of the arbitrariness in pinching

In the examples discussed so far in this section, the uncertain inputs were pinched in particular ways. In the sensitivity study summarized in Figure 7, for instance, we pinched the input A to a precise probability distribution, but there are many distributions we might have used to replace the uncertain number. In some situations, there may be an obvious candidate for the hypothetical input to which the uncertain number should be pinched. This may be the "best" estimate, whether a distribution or point value, or a central estimate in some sense, or the nominal estimate used in a preliminary assessment. In many cases, however, the pinching will necessarily be arbitrary because there will be many possible choices to replace an uncertain number. In each of these cases so far, the pinching used was not the only one possible. Other potential pinchings will typically yield different calculated reductions of overall uncertainty. What effect does the choice of the distribution or point have on the sensitivity estimate?

In study 1 (illustrated in the middle panel of Figure 7), pinching the uncertain number A to the uniform distribution between 4.5 and 5.5 yielded a reduction of uncertainty of 47.1%. How much bigger or smaller might this percentage have been had we pinched A to some other probability distribution consistent with what is known about A? In other words, if we replace the p-box with other probability distributions that lie within the p-box, how different might this percentage be? Using a simulation study that randomly replaces A with a uniform distribution whose endpoints are consistent with the uncertain number and redoing the reduction calculation, one can find that the possible reductions in uncertainty are always within the range [46.2, 50.4]%. Similar simulations can be constructed to explore the ranges of uncertainty reductions for other variables in the other sensitivity studies. The table below summarizes the results from such simulations for the eight studies depicted in Figure 7 though Figure 10. Notice that the point pinchings in studies 6 and 7 do not create any variation in the percent reductions observed. The reason for this is that, for the function of addition, the breadth of the output is totally determined by the breadth of the unpinched input whenever the other input is a scalar. (This would not have been the case if the function had been multiplication rather than

addition.) There was no variation in the percent reduction that resulted from pinching the uncertain number *B* in studies 2 and 4 either. In this case, however, the reason was that the uncertain number was defined as a class of distribution functions that all had the same variance, so all the pinchings were different only in their means. Consequently, the reduction in the breadth of the sum was always the same.

Sensitivity study	*Pinched input*	*Range of percent reductions*	*Least pinching*	*Greatest pinching*
1	*A*	[46.2, 50.4]	*A* = uniform(4,6)	*A* = 5
2	*B*	47.2	–	–
3	*A*	[0, 65.6]	*A* = uniform(4,6)	*A* = 5
4	*B*	32.8	–	–
5	(dependence)	[30.4, 32.5]	Clayton ($r = -0.64$)	comonotonic
6	*A*	50.4	–	–
7	*B*	52.3	–	–
8	*A*	32.3	–	–

The last two columns of the table give the assignments that led to the smallest and largest reductions. For instance, pinching *A* to the uniform distribution between 4 and 6 resulted in a 46.2% reduction in uncertainty compared to the original uncertain number, and pinching it to the constant value of 5 yielded a reduction in uncertainty of 50.4%. In study 5, the Fréchet non-assumption about dependence was pinched to an assumption of independence. The simulation showed that assuming that *A* and *B* were monotone functions of each other, either positively or negatively, the reduction in uncertainty would be largest. This relationship is called comonotonicity (Ferson et al. 2004). The simulation achieved the smallest reduction in uncertainty when the relationship between *A* and *B* was described by an intermediate dependence function described by Clayton (Nelsen 1999; Ferson et al. 2004) with correlation coefficient $r = -0.64$.

Notice that the range of variation in percent reduction of uncertainty is often small or zero, but that it can also be rather large. For instance, in study 3 (which is illustrated in the second panel of Figure 8), the reduction ranges from zero to almost 66%. The lower bound in this case comes from pinching the uncertain number *A* to a uniform distribution between 4 and 6. It might be initially surprising that this same pinching was associated with a change of only a few points in study 1 but, in study 3, has totally erased any reduction in uncertainty. The reason for this is that the stricture of independence that governed and constrained the results in study 1 was absent from study 3. Because the range of the pinched input *A* is the same as that of the original uncertain number, the best possible bounds on the convolution without making any assumption about dependence (the Fréchet case) are just the same as they were for the baseline case, yielding possibly no reduction of uncertainty. This is not always true; if the function had been multiplication rather than addition, the smallest possible reduction in uncertainty for the Fréchet case would have been small but bigger than zero. Likewise, the simulation result for study 4 shows that the range of reductions is not always wide under the Fréchet assumption. In this study, it is a constant. In study 8, there is no uncertainty about the

percent reduction because there is only one possible way to pinch the aleatory uncertainty from *A* while retaining its epistemic uncertainty.

For several of these numerical examples, the chore of discovering the extremal reductions is relatively simple because the uncertain distributions represent sets of distribution functions that can be parameterized by one or two dimensions. For instance, input *A* in Figure 7 through Figure 9 consists of all uniform numbers between a minimum ranging on [4, 5] and a maximum ranging on [5, 6]. Thus, the simulation only needs to select minima and maxima randomly from these two ranges to construct all possible distributions consistent with the uncertain number. Likewise, the input *B* for the same studies represents all unit-variance normal distributions with means on the interval [8, 9], so the simulation need only select one mean at random. The problem is not much harder for the class of triangular distributions used for input *B* in Figure 10 as only three values need to be selected.

However, in general, the simulation problem is *much* harder than these simple cases would suggest. Consider, for example, the distribution-free input *A* in Figure 10 defined by a minimum, maximum and mean. In this case, there is no family of input distributions that can be parameterized in a small number of dimensions. In fact, the set of distribution functions corresponding to this uncertain number is infinite-dimensional, which means that you would in theory need an infinite number of parameters to specify all the distributions in the class represented by this uncertain number. (In practice, this just means that you would have to specify a *graph* of the distribution that shows the cumulative probability for each possible value of the abscissa.) The consequence of this complexity is that we cannot expect any simulation strategy to reliably discover the full extent of the class of distributions.

The same problem afflicts the calculation of the range of reductions that are possible with different dependence patterns. There are infinitely many possible dependence relations between *A* and *B*, and there is no way to use a simulation approach to properly sample their diversity. The range of reductions we found for study 5 is not guaranteed to be comprehensive (Ferson and Hajagos 2006; Ferson et al. 2004). We obtained this range by varying correlation coefficients from −1 to +1 for a handful of dependence families (copulas). Methods need to be developed that can compute rigorous bounds on uncertainty reductions for this situation.

Sometimes the problem of computing the range of reduction is straightforward and computationally easy. In such cases, the results are comprehensive and reveal how much or little one can trust the estimate of the sensitivity. In other cases, however, depending on whether the uncertainty involves the form of dependency or the family of distributions, the problem is considerably harder. It will often still be possible to conduct simulations and obtain so-called "inner bounds" on the range of reductions due to the arbitrariness of the pinchings. Such ranges are not guaranteed to enclose all possible reductions, but they still might be usefully informative.

4.2 Different measures of uncertainty

Aside from the arbitrariness considered in the previous section from the values or distributions to which the uncertain numbers are pinched, there is also an element of

arbitrariness—or perhaps we should say *flexibility*—in how the uncertainty is measured. This report has elected to use breadth, i.e., the area within the p-box or between the cumulative plausibility and cumulative belief functions of a Dempster-Shafer structure, as the unc( ) function. Many researchers in probabilistic sensitivity analysis have suggested that variance is the natural way to measure uncertainty of probability distributions (e.g., Saltelli 2003; Saltelli and Tarantola 2002). Others have argued that entropy is the best such measure (e.g., Jaynes 2003).

We believe that there are many possible measures of uncertainty, both in (precise) probabilistic models and in imprecise probability models such as DST and PBA. Besides the breadth, variance and entropy, we might also consider the simple range or the interquartile range. Hall (2006) suggested the use of relative entropy, which is the Kullback distance* between the distribution of the model output and each input distribution.

There is an important practical issue to consider when defining unc( ): scalar measures are easier to work with. Variance and entropy for probability distributions are scalars, but uncertain numbers—as we have defined them—in general represent *classes* of probability distributions. This suggests that "the" variance or entropy for an uncertain number will typically be a set of scalar values rather than a point value. For example, because the formula for the entropy of a continuous uniform distribution with width w is $\log_2(w)$, the entropy for the uncertain number A in the top panel of Figure 7 is the range $(-\infty, 1]$. The upper bound is the entropy of the uniform distribution ranging between 4 and 6. The lower bound represents the entropy of a degenerate Dirac distribution at 5 which is also a member of the class. But just because there is epistemic uncertainty present does not mean that the variance or entropy measures will be intervals. For instance, the uncertain number B in the top panel of Figure 7 has unit variance, or rather each distribution that is consistent with it has unit variance. Therefore, its variance must simply be one. Likewise, because all the distributions in this p-box are normals and because the formula for the entropy of a normal distribution with parametric standard deviation σ is $\log_2\left(\sigma\sqrt{2\pi e}\right)$, the entropy for B is the scalar number 2.047. This happens because all of the distributions in this class have the same shape and dispersion. Thus the p-box has constant entropy even though it would seem to have significant epistemic uncertainty.

Several researchers (e.g., Klir 2006; Ross 2005; Hall 2006) have studied the issues surrounding how a scalar measure of overall uncertainty might be defined in the general

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*The Kullback distance between a probability distribution P and a theoretical probability distribution Q is, in the discrete formulation,

$$\sum_i P(i)\log_2\frac{P(i)}{Q(i)}.$$

This distance is interpreted as the expected extra message length per datum needed to communicate a value when an encoding is optimal for a given distribution Q, rather than using an encoding based on the true distribution P.

context of imprecise probabilities. Information entropy is widely regarded as *the* measure of randomness (Jaynes 2003), and some have suggested that the maximum entropy of any distribution consistent with an uncertain number might be a suitable measure of overall uncertainty for an uncertain number too (e.g., Klir 2006). One can also imagine defining overall uncertainty as the largest variance of any distribution consistent with an uncertain number, or the range of variances, or the width of the range of variances, i.e., the difference between the largest and smallest possible variance.

Each possible definition of unc( ) measures a particular feature of the uncertain number that might be appropriate in some particular setting or application. For instance, the breadth of any precise probability distribution is always zero, so this measure would not be useful when there is no epistemic uncertainty at play. The first column of Figure 11 illustrates this. It shows a spike representing the Dirac delta distribution of a scalar number, a distribution that looks like an exponential distribution, and a uniform distribution, which all have the same breadth of zero even though they are quite dissimilar from one another. Likewise, the middle column depicts a uniform distribution, a p-box and an interval that all have the same largest possible entropy (which is that of the uniform distribution from the lower, left corner to the upper, right corner). The last column points out that range as an overall measure of uncertainty would not distinguish any distributions having the same support, or even the same width of their support.

There are many measures of overall uncertainty that an analyst might employ, and no single measure seems to be best for all uses. Moreover, it seems clear that any scalar measure of overall uncertainty must be profoundly limited in one way or another. After all, it is a scalar measure attempting to characterize something that an entire interval, distribution, p-box or Dempster-Shafer structure is expressing. Nevertheless, as long as the analyst is aware of the properties of the measure employed to characterize overall uncertainty, its use in defining measures of sensitivity seem to be sound.

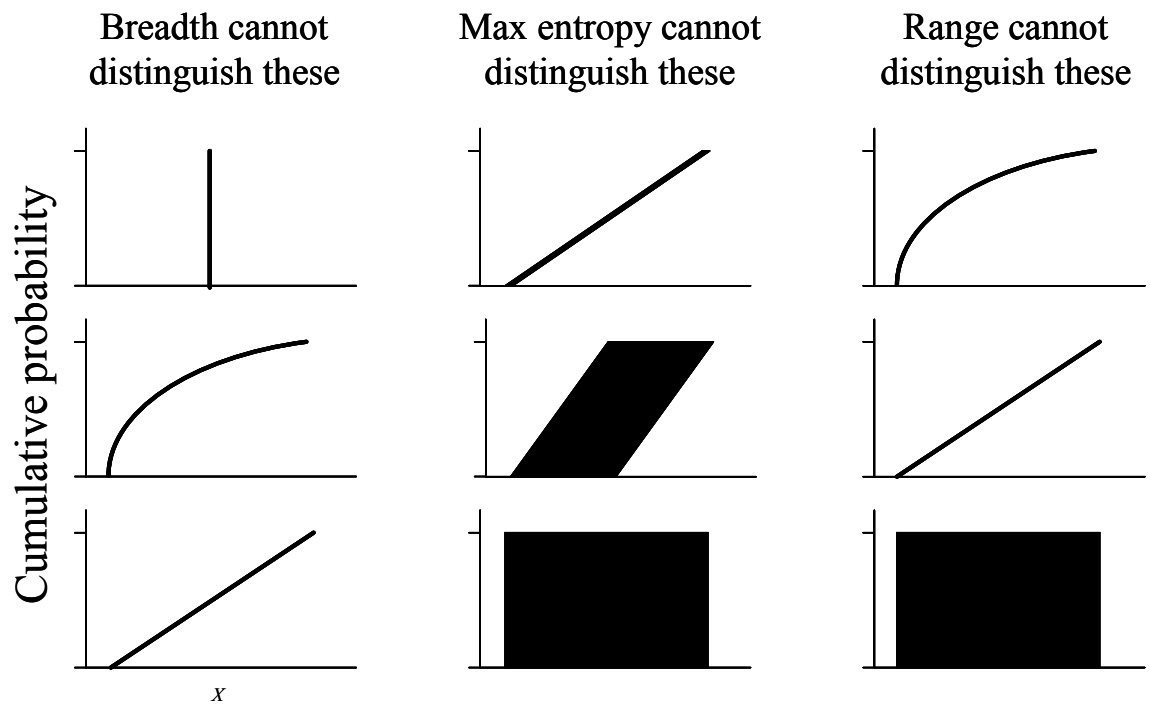

Figure 11. Uncertain numbers that three common measures of overall uncertainty would not distinguish.

5 Case study: dike revetment

This section considers a small case study involving a dike's reliability assessment to illustrate the methods discussed in this report for sensitivity analysis with uncertain numbers.

Figure 12 depicts a schematic diagram of a dike and its revetment, which is the seaward facing of masonry that protects it from the surf. The reliability of the structure depends in part on the density and thickness of the revetment and its angle of inclination. A reliability formula of strength minus stress for such a dike was given by Hussarts et al. (2000) and modified by Hall and Lawry (2001) as

$$Z = \Delta D - \frac{H \tan(\alpha)}{\cos(\alpha) M \sqrt{s}}$$

where Δ is the relative density of the revetment blocks, D is their thickness, H is the significant wave height (which is the defined to be the average height of the highest one third of waves), α is the slope of the revetment, s is the offshore peak wave steepness, and M is the model parameter, which is a factor introduced to represent the analyst's uncertainty about the model itself. If the reliability function Z is surely positive, then the dike is inferred to be reliable. If, however, Z takes on negative values, then the structure is not sufficiently strong to withstand the stresses from the sea. The probability that Z is positive is taken to be the reliability of the structure.

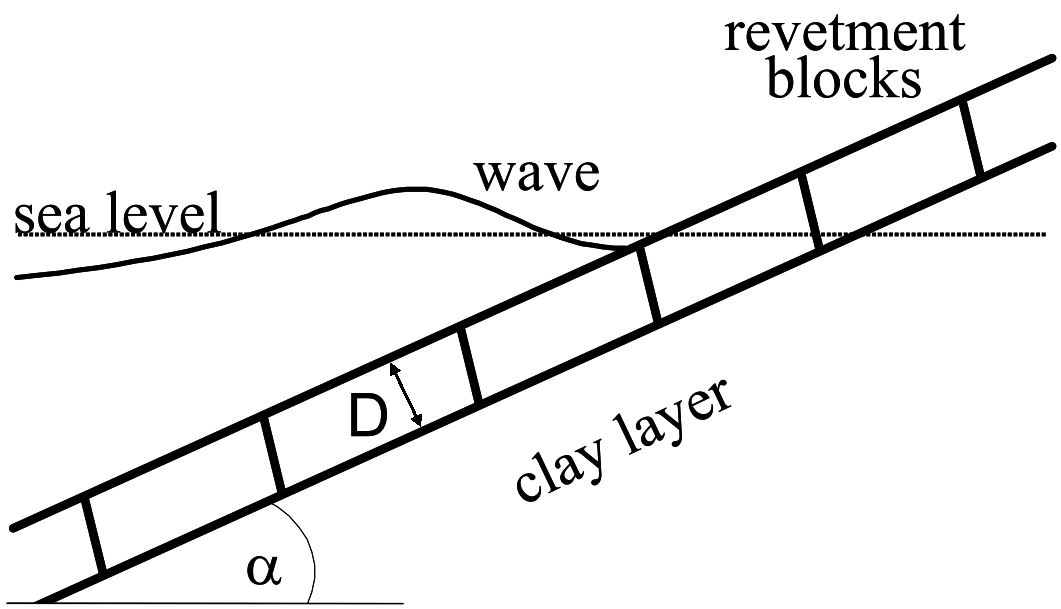

Figure 12. Dike revetment (redrawn from Hussarts et al. 1999).

For this example, we will suppose that

$\Delta \in [1.60, 1.65]$,

$D \in [0.68, 0.72]$ meters,

$\alpha \in$ arc tangent([0.32, 0.34]) = [0.309, 0.328] radians,

$M \in [3.0, 5.2]$,

$H \sim$ Weibull(scale = [1.2,1.5] meters, shape = [10,12]), and

$s \sim$ normal(mean = [0.039,0.041], stdev = [0.005,0.006]).

For the purposes of this example, we've modified these assignments from those used by Hall and Lawry (2001). Some of the parameters have interval uncertainty; for Δ, D, α and M we are only specifying bounds on the possible values. The parameters H and s, on the other hand, are known to vary stochastically because they are functions of weather, seasonality and the chance of storm events. Their probability distributions are imprecisely characterized however. In this case, the analyst feels confident about specifying the shapes of their distributions, but can only bound the distributional parameters.

In the following subsections, we review the traditional probabilistic approach to propagating the expressed uncertainty, the generalization with uncertain numbers using the new methods of DST and PBA, the calculation of local derivatives, and a sensitivity analysis via pinching studies.

5.1 Traditional probabilistic approach

A traditional reliability assessment such as might be implemented in a Monte Carlo simulation would require the selection of (precise) probability distributions for each input. For uncertainty expressed as intervals it is common to select a uniform distribution over the given range. For the two distributions H and s, a traditional approach might have used the best estimates of the parameters. For this example, we shall assume these would be the midpoints of the respective intervals. The inputs for a Monte Carlo simulation would therefore be

Δ ~ uniform(1.60, 1.65),

D ~ uniform(0.68 meters, 0.72 meters),

α ~ arc tangent(uniform(0.32, 0.34)),

M ~ uniform(3.0, 5.2),

H ~ Weibull(scale = 1.35 meters, shape = 11), and

s ~ normal(mean = 0.04, stdev = 0.0055).

These inputs are depicted as cumulative distribution functions in Figure 13. Had any of the inputs not been specified as to distribution family, a traditional probabilistic assessment might have used the maximum entropy criterion (Jaynes 2003) to select the distribution from the available information.

The result of convolving these distributions together in a Monte Carlo simulation based on 10 million replications is the distribution depicted as a dotted curve in Figure 14. This calculation suggests the chance that Z is negative is about 0.0000141. Many replications were needed to obtain an accurate measure of this risk, because it is in the far left tail of the distribution.

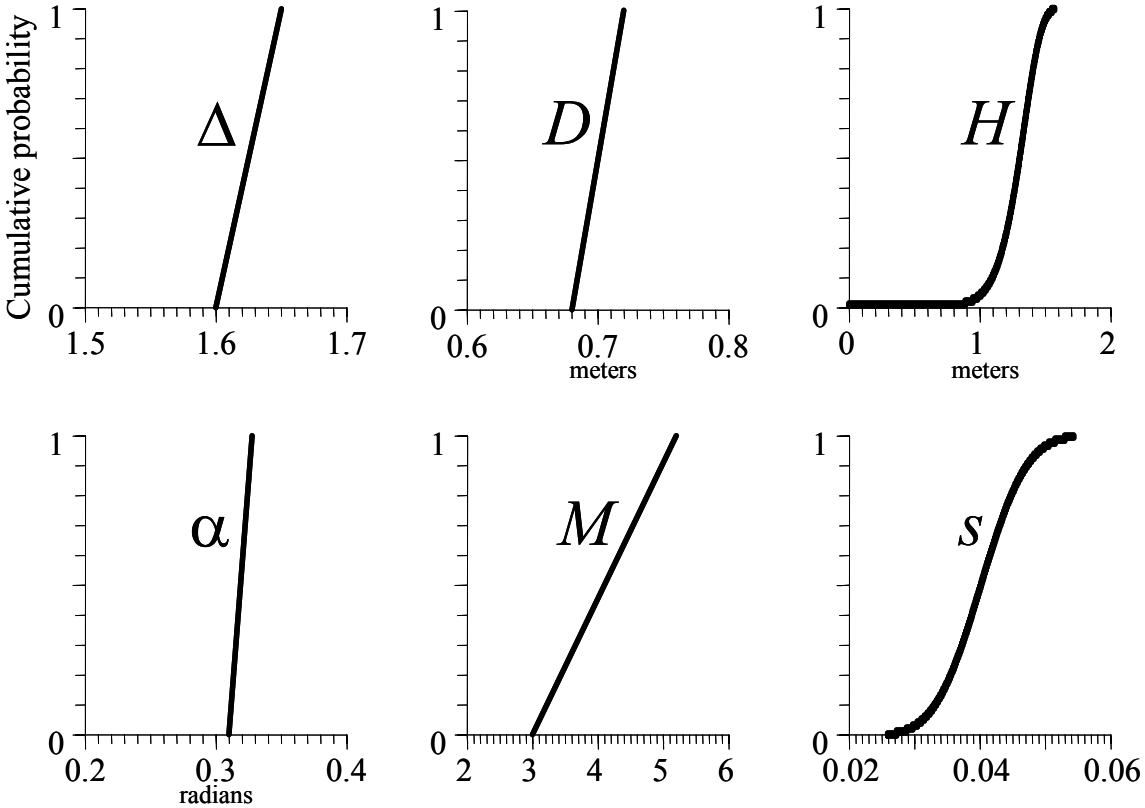

Figure 13. Probability distribution inputs for the dike reliability assessment.

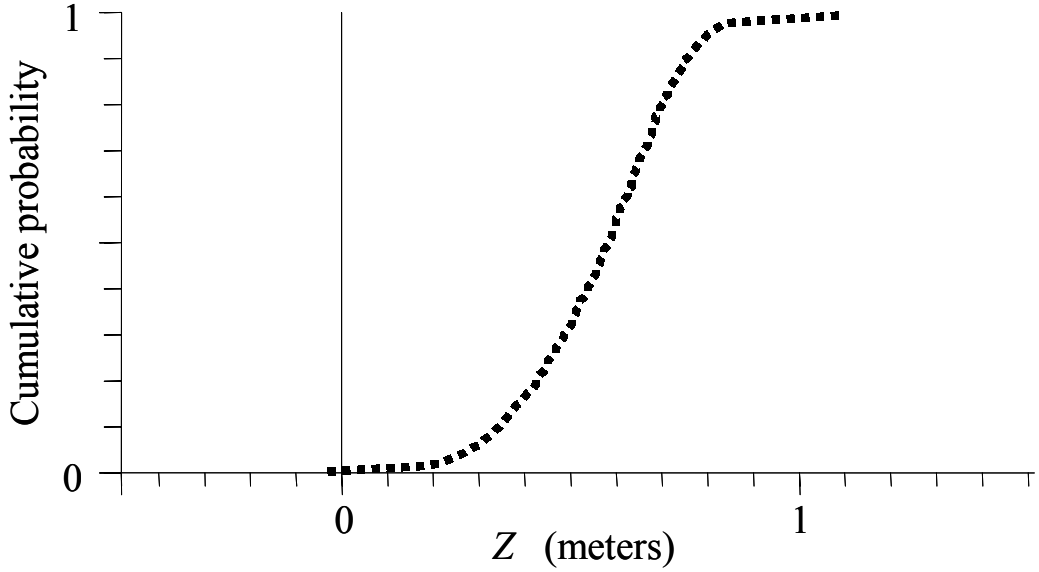

Figure 14. Distribution of reliability *Z* from a traditional Monte Carlo analysis.

5.2 What-if sensitivity analysis with DST and PBA

Because the inputs are assumed to be independent of one another, the function Z is easy* to compute using PBA and DST (Yager 1986; Williamson and Downs 1990; Ferson et al. 2003; Ferson and Hajagos 2004).

All six uncertain inputs are depicted in Figure 15. The upper bound of each p-box is the cumulative plausibility function for the variable and its lower bound is its cumulative belief function. Four of the inputs are simple intervals, which are degenerate p-boxes and Dempster-Shafer structures having only one focal element. The other two inputs, H and s, are specified as known distribution shapes with imperfectly known parameters given as intervals. Ferson et al. (2003) detailed how such information about H and s can be summarized as p-boxes, and how they can, via a discretization of horizontal slicing be decomposed into Dempster-Shafer structures.

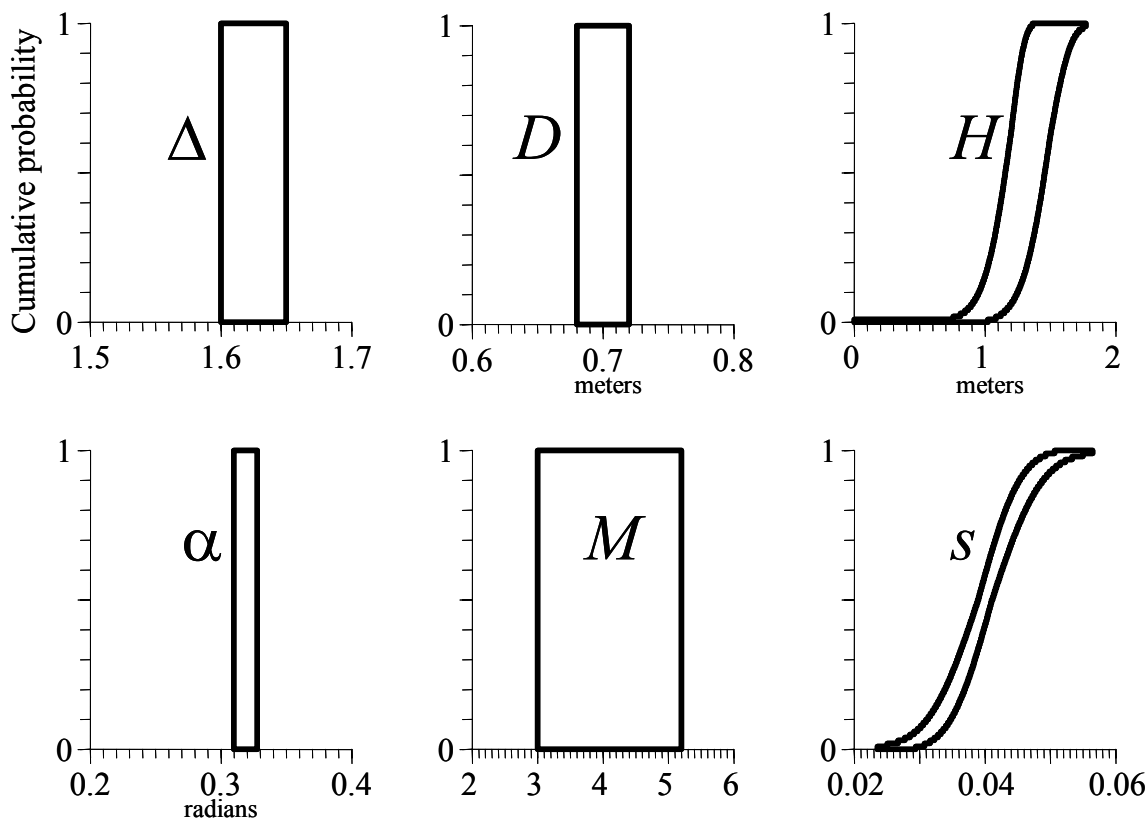

Figure 15. Uncertain number inputs for the dike reliability assessment.

*Interestingly, although the variable α is uncertain and repeated in the expression for Z, it does not lead to an inflation of the uncertainty in this particular example. The reason is that the function is monotone increasing over the range of this variable's uncertainty.

The Dempster-Shafer structure for the variable Δ is a single interval-mass pair {([1.60, 1.65], 1.0)}. Likewise, the Dempster-Shafer structures for *D*, *M* and α are {([0.68, 0.72] meters, 1.0)}, {([3.0, 5.2], 1.0)}, and {([0.309, 0.328] radians, 1.0)} respectively. The Dempster-Shafer structures for the other two inputs must be obtained by discretizing their p-boxes. Using one hundred equiprobability levels, these discretizations yield {([0, 1.02], 0.01), ([0.76, 1.08], 0.01), ([0.81, 1.12], 0.01), …, ([1.34, 1.75], 0.01), ([1.36, 1.77], 0.01)} for the input *H*, and {([0.0235, 0.0294], 0.01), ([0.0250, 0.0307], 0.01), …, ([0.0493, 0.0550], 0.01), ([0.0506, 0.0565], 0.01)} for the input *s*.

The mechanics of the calculation with these uncertain numbers are the same in both PBA and DST and involve the creation of a Cartesian product between all the focal elements of all six inputs. Ordinarily, this would generate a six-dimensional Cartesian product, but, in this case, because four of these inputs have only a single focal element, the Cartesian product ends up being simple enough to depict as a two-dimensional matrix of size 100×100 consisting of the combinations of all possible pairs from the discretization of inputs *H* and *s*. The resulting Cartesian product is displayed below.

***Z*(Δ,*D*,*M*,α,*H*,*s*)** **Independent**	[0, 1.02] 0.01	[0.76, 1.08] 0.01	[0.81, 1.12] 0.01	…	[1.34, 1.75] 0.01	[1.36, 1.77] 0.01
[0.0235, 0.0294] 0.01	[0.290,1.19] 0.0001	[0.242, 0.903] 0.0001	[0.213, 0.882] 0.0001		[−0.275, 0.682] 0.0001	[−0.294, 0.675] 0.0001
[0.0250, 0.0307] 0.01	[0.314, 1.19] 0.0001	[0.268, 0.909] 0.0001	[0.239, 0.889] 0.0001		[−0.233, 0.693] 0.0001	[−0.252, 0.686] 0.0001
⋮				⋱		
[0.0493, 0.0550] 0.01	[0.536, 1.19] 0.0001	0.503, 0.980] 0.0001	[0.483, 0.965] 0.0001		[0.145, 0.818] 0.0001	[0.132, 0.813] 0.0001
[0.0506, 0.0565] 0.01	[0.544, 1.19] 0.0001	[0.511, 0.983] 0.0001	[0.491, 0.968] 0.0001		[0.158, 0.823] 0.0001	[0.145, 0.818] 0.0001

The margins of this Cartesian product are the discretization of the uncertain number *H* (displayed as the top row) and the discretization of the uncertain number *s* (displayed as the leftmost column). Each element of the margins consists the focal element (displayed within square brackets) and its associated mass (displayed below each interval), which is always 0.01 in this case study. Each cell within the Cartesian product is also an interval together with its associated mass. Because *s* and *H* are independent, the mass associated with each of these 10,000 cells is 0.01×0.01 = 0.0001, so that their sum over all the cells is exactly unity. These masses are written as the lower lines of the cells. On the upper line of each cell is the interval, that is, the focal element of the output *Z*, with which the

mass is associated. Each interval is computed by applying the reliability function to the intervals for that cell

$$Z_{i,j} = \Delta D - \frac{H_i \tan(\alpha)}{\cos(\alpha) M \sqrt{s_j}}$$

where H_i and s_j are the ith and j^{th} focal elements from discretized Dempster-Shafer structures for H and s respectively. For instance, the first focal elements from H and s are H_1 = [0, 1.02] and s_1 = [0.0235, 0.0294] respectively, so the first entry in the upper, leftmost cell of the Cartesian product is

$$\begin{aligned} Z_{1,1} &= \Delta D - \frac{H_1 \tan(\alpha)}{\cos(\alpha) M \sqrt{s_1}} \\ &= [1.60, 1.65] \times [0.68, 0.72] \text{ meters} - \frac{[0, 1.02] \times \tan([0.309, 0.328] \text{ radians})}{\cos([0.309, 0.328] \text{ radians}) \times [3.0, 5.2] \times \sqrt{[0.0235, 0.294]}} \\ &= [0.290, 1.19] \text{ meters,} \end{aligned}$$

which is computed by straightforward application of the elementary rules of interval arithmetic (Moore 1966). By convention, the intervals are expressed with outward-directed rounding, which is to say, the lower bounds are always rounded down and the upper bounds are rounded up. Each of the computed intervals has units of meters.

These 10,000 focal elements are reassembled into the p-box displayed as the (solid) bounds on the reliability distribution for Z in Figure 16. These bounds reveal the chance that Z is negative is within the interval [0, 0.044], which is to say that it is surely less than about one in twenty three. The breadth of this interval probability reflects the epistemic uncertainty embodied in the uncertain numbers given for the six input variables. For comparison, the distribution from the prior Monte Carlo simulation is also shown in the same figure as a dotted curve.

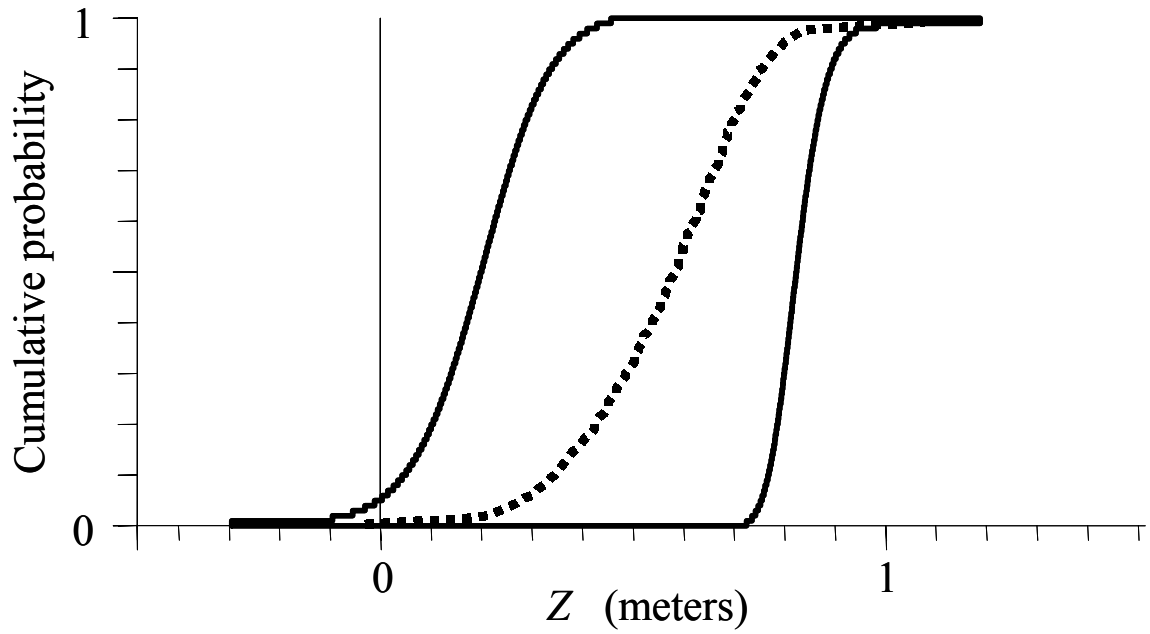

Figure 16. Bounding (solid) and Monte Carlo (dotted) estimates for the distribution of reliability *Z*.

The random variables in this expression are assumed to be independent, although we could have assumed otherwise, or even made no assumption about their dependence, and used the same approach. PBA supports convolutions under several models of dependence (Ferson et al. 2004). For instance, even if we assumed nothing about the stochastic dependencies among the input parameters, we could have concluded that the risk that Z could be negative is surely smaller than roughly 0.24, or a little less than one in four.

We note that the difference between the result from the traditional probabilistic analysis and those from the new methods is substantial. Using the more comprehensive new methods, the risk of dike failure is seen to be over 3,600 times larger than was originally estimated, and much larger still if the independence assumptions are relaxed. Large discrepancies between the traditional and new analyses are common, especially for tail risk calculations.

5.3 Second-order probabilistic analysis

How would the results of a second-order or "two-dimensional" Monte Carlo analysis compare with those seen in sensitivity study using p-boxes and Dempster-Shafer structures? To answer this question, we also conducted a second-order Monte Carlo simulation for the dike revetment case study.

The simulation used 10,000 Monte Carlo replications in the outer loop and a semi-analytical convolution (Kaplan 1981) using a 1000 discretization levels in the inner loop. Like the traditional, one-dimensional probabilistic approach described in section 5.1, this simulation used uniform distributions to model the epistemic uncertainty in Δ, D and M.

The epistemic uncertainty about the mean and standard deviation of the normal distribution for s and the scale and shape parameters for the Weibull distribution for H were also modeled with uniform distributions. By taking arc tangents of a uniformly distributed deviate, we modeled the epistemic uncertainty about α as a tangent-uniform distribution. The selection of the values of these eight variables occurred in the outer loop of the simulation and the values were then held constant in the inner loop. The inner-loop convolution between the distributions for s and H used a discrete approximation using one million values (1000×1000 discretization levels). All variables were assumed to be independent of one another. All distribution parameters were also assumed to be independent within and across distributions.

The results of this analysis were 10,000 possible distributions for the Z variable. The mean cumulative probability at $Z = 0$ was $0.000149 = 1.49\times10^{-4}$. The largest such cumulative probability for any of the 10,000 was only 0.008. The result of DST and PBA suggest that the largest risk of Z being negative could be almost 300 times larger than the mean risk, and over 5 times larger than the second-order probabilistic analysis ever saw in 10,000 replicates. This case study illustrates that both traditional and second-order probabilistic analyses depict much smaller risks than are revealed by the new methods of DST and PBA. The discrepancy is so stark that one might suspect that the new methods are somehow inappropriately inflating the risks. It is easy to check this is not the case, however. The dike revetment case study is simple enough to confirm the results of the DST and PBA calculation by inspection. The extreme risk occurs when α, the standard deviation of the normal distribution, and scale (first) parameter of the Weibull distribution are at the high ends of their respective ranges, and all the other parameters are at the low ends of their ranges. Given that these ranges were intended to represent epistemic uncertainty rather than aleatory uncertainties, it is presumably entirely plausible that such a combination of values (or at least values close to these) might actually occur. If this is the case, then it is clear that both the traditional and the second-order assessments are understating the plausible risks. Because they do not use assumptions that uncertain parameters vary according to a uniform (or any other) distribution or that variables and parameters are independent (or have any other dependency) unless such assumptions are warranted, PBA and DST can reveal the full uncertainty about the risks for the dike revetment.

5.4 Local derivatives

The partial derivatives of the Z reliability function with respect to its six inputs are

$$\frac{\partial Z}{\partial \Delta} = D,$$

$$\frac{\partial Z}{\partial D} = \Delta,$$

$$\frac{\partial Z}{\partial \alpha} = \frac{-H\left(1+\sin^2(\alpha)\right)}{\cos^3(\alpha)M\sqrt{s}},$$

$$\frac{\partial Z}{\partial M} = \frac{H\tan(\alpha)}{\cos(\alpha)M^2\sqrt{s}},$$

$$\frac{\partial Z}{\partial H} = \frac{-\tan(\alpha)}{\cos(\alpha)M\sqrt{s}}, \text{ and}$$

$$\frac{\partial Z}{\partial s} = \frac{H\tan(\alpha)}{2\cos(\alpha)Ms^{3/2}}.$$

These formulas can be evaluated using the uncertain number estimates for the inputs that were depicted in Figure 15 using PBA or DST. The results of these calculations are the sensitivity coefficients displayed in Figure 17. Even though the overall uncertainty is large, the results are quite informative. In particular, they are all sign-determinate, although a few extend to the origin in their tails. A ranking that seems to survive the large uncertainty is, from largest positive to largest negative, s, D, Δ, M, H, α. The largest sensitivity is to the peak wave steepness s. It is almost surely larger than any of the positive sensitivities, and could be much larger in absolute magnitude than the sensitivity to the dike's angle of incline α, which is the second largest in magnitude. In fact, most of the generalized sensitivity coefficients don't overlap much, except at their extreme tails. The peak wave steepness presumably represents the effects of weather and tidal cycles and is probably not under the engineer's control, unless other features of the harbor or near-shore environment could be modified. The angle of incline α and the thickness of the revetment blocks D, however, are perhaps the most suitable variables for manipulation to affect the magnitude of Z. It is interesting and important that the sensitivity to the model uncertainty factor M is fairly small compared to the other inputs. This suggests that the doubt it represents about the structure of the model may be of moderately small significance. The table below summarizes some of the statistics of these sensitivity coefficients.

Sensitivities

Coefficient	Range	Mean	Median	Variance
$\partial Z/\partial\Delta$	[0.68, 0.72]	[0.68, 0.72]	[0.68, 0.72]	[0, 0.0004]
$\partial Z/\partial D$	[1.6, 1.65]	[1.6, 1.65]	[1.6, 1.65]	[0, 0.000625]
$\partial Z/\partial\alpha$	[–5.01, 0]	[–3.20, –1.37]	[–3.22, –1.38]	[0.0660,0.941]
$\partial Z/\partial M$	[0, 0.461]	[0.070,0.295]	[0.0700,0.296]	[0.000196,0.0142]
$\partial Z/\partial H$	[–0.780, –0.272]	[–0.615, –0.319]	[–0.607, –0.319]	[0.00246, 0.0286]
$\partial Z/\partial s$	[0, 29.359]	[4.5004, 11.902]	[4.3544, 11.355]	[0.57662, 26.852]

Except for the second and penultimate lines of this table, the intervals have units of meters, or square meters for the variances.

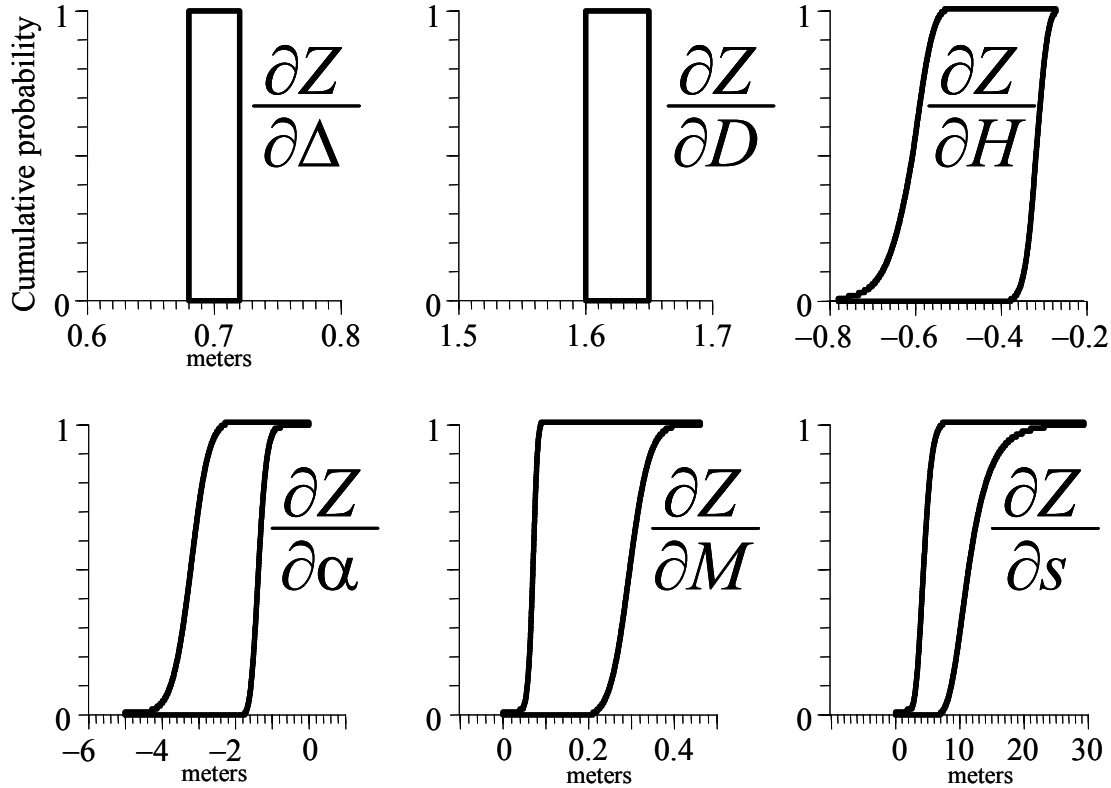

Figure 17. Local sensitivities of the reliability function to its uncertain inputs.

5.5 Reductions in uncertainty by pinching

The local derivatives computed in the previous section above allow analysts to discern which variables might be manipulated to obtain the largest impacts on the reliability Z. This section considers the calculation of potential reductions in uncertainty by pinching the uncertain inputs to remove epistemic uncertainty. This analysis reveals which input variables should be the focus of future empirical effort so that our overall uncertainty about Z might be minimized. The results of applying the pinching strategies described in section 4 are displayed in the summary table below.

	Percent reduction of uncertainty	
Input	Nominal pinching	All possible pinchings
Δ	5.5	[4.7, 5.7]
D	10.0	[9.2, 11.0]
M	53.0	[41.0, 60.0]
α	6.5	[3.8, 9.1]
H	23.0	[15.0, 30.0]
s	3.6	[2.0, 5.2]

The middle column shows the percentage reduction in uncertainty about Z that was obtained by replacing the uncertain number named in the first column with a nominal value. This nominal value was either a scalar number or the precise probability distribution used in the traditional probabilistic assessment. For the four inputs characterized by intervals (Δ, D, M and α), the uncertain number was pinched to the midpoint of its interval. In the case of the input H, the variable was pinched to a Weibull distribution with scale parameter 1.35 meters and shape parameter 11. For s, the nominal pinching was to a normal distribution with mean 0.04 and standard deviation 0.0055.

The third column of the table shows the range of uncertainty reductions that could be obtained from all possible pinchings of the given input. In the cases of the four inputs that were given as intervals, the simulations solved constrained optimizations to discover the range of reductions in uncertainty that could be obtained by pinching the input to any probability distribution or scalar that was consistent with its uncertain number. A scalar is consistent with an interval if it lies within the interval. A probability distribution is consistent with an interval if its support lies entirely within the interval. For the inputs H and s, the uncertain numbers were specified as having Weibull and normal distributions respectively, which were parameterized by intervals. For these variables, the simulations sought the range of reductions that could be obtained by pinching the input to a particular distribution of the specified shape such that the parameters of this distribution were consistent with the parameters' specified limits. The constraint analyses for this case study were easy to do, although such analyses in general could be fairly complex (Jaulin 2001; Walley 1991).

The results of these pinching analyses suggest which uncertain numbers deserve focus in future empirical studies to reduce the overall epistemic uncertainty about the reliability Z. The percent reductions from the nominal pinchings suggest that M would be best to study, followed by H, D, α, Δ and s, in that order. The ordering among M, H and D is only reinforced by the reduction ranges, which agree with it despite the large uncertainties that are present. In other words, the rankings would not change, *no matter how the pinching might be done*. This gives us considerable confidence in the surety of this finding. The rankings among α, Δ and s, however, are confused when we look at the reduction ranges because their ranges overlap broadly.

To reduce the overall epistemic uncertainty about the dike's reliability, it would be most effective to study the value M which is the model uncertainty factor that the analysts introduced to account for their own discomfort at what they felt was an overly precise model. It is especially interesting that this analysis reveals that M contributes a lot to epistemic uncertainty about Z, particularly because, as we saw in the previous section, altering M would have little effect on the magnitude of Z itself. This discrepancy is not really counterintuitive in any way, but probabilistic sensitivity analyses that do not distinguish epistemic and aleatory uncertainty in the way we have here might not have recognized that the model can be epistemically sensitive to the input M without its being a useful focus of management or control.

6 The anova myth

This brief section criticizes a view that is becoming very common in probabilistic sensitivity analysis that variance is the appropriate measure of uncertainty.

The importance of a parameter in an uncertainty analysis is the degree to which its uncertainty contributes to the uncertainty of the output. This notion has been called "sensitivity" by Saltelli (2003) and "influence" by Andres (2003). Andres (2003) asserted that there can be at most 10 influential variables if we define an 'influential variable' as one that contributes no less than 10% of the uncertainty. The idea is that simply that no more than 10 variables could have more than 10% of a total which seems to be tautological. This claim depends on the idea that what's important about uncertainty is the *variance*, and the properties of the variance, particularly, the fact that the variance can be partitioned into components.

Of course, variance is not the only measure of uncertainty. In fact, very few of the reasonable measures of uncertainty actually behave in this way. Consider, for instance, a parameter's *range*, i.e., the difference between the largest and smallest possible values. It is obviously another measure of uncertainty and it is commonly used for this purpose. It does not partition like variance does in the way needed for Andres' computational strategy to work. For instance, consider the following simple uncertainty analysis conducted with interval analysis. Suppose there are 3 parameters to be multiplied together. And suppose for this example that the uncertainty about these parameters is such that each ranges on the interval [0, 2]. Obviously, the range of the product is just the interval $[0, 2^3] = [0, 8]$. Replacing any one of the parameters by its midpoint would reduce the range of the product by half to [0, 4]. If we measured the importance of a parameter by the reduction in the width of these intervals, we would say that the importance of each of the three parameters was 50%. Now suppose that there are many such parameters to be multiplied. No matter how many parameters there are, the importance of each is 50%. This little example shows that uncertainty, as distinguished from variance, need not partition in the way Andres suggests it must. This does not depend on the uncertainties of the inputs being similar in magnitude. If the importance of a parameter is measured by the percent reduction of uncertainty associated with removing the parameter from the model (or pinching it to its mean or some other scalar value), uncertainty analysts often observe that the sum of these importance values for the various parameters add up to something larger than 100%. We argue that one cannot overcome this example by claiming that range is a somehow odd or unreasonable measure of uncertainty. The underlying truth is that uncertainty is a complicated and multivariate notion that cannot really be captured completely by the fairly simplistic notion of variance.

Andres (2003; cf. 2004) offered as an example the Canadian House of Commons, which is the lower house of the Parliament of Canada, and argued that the number of influential members of parliament were necessarily few. Andres' legislative example may itself provide a counterargument to his idea that influential variables are necessarily few. In some legislative bodies, notably the United States Senate, the rules enable members to obstruct legislative work with a filibuster, by which any senator can prevent

a vote on any question. Senators jealously guard their right of filibuster, precisely because it is so empowering to each of them. It enables a senator to derail any legislation that might come under consideration by the chamber. In this sense, it is a right of veto that can only be denied by a three-fifths vote of cloture (to limit debate). Thus, if we measure the importance of a senator by his or her ability to kill legislation, we must admit that each of the hundred senators truly is influential, or at least could be.

Even in the context of the parliamentary metaphor, we can see clearly that 'what is important' is much more complicated than what can be represented in the narrow statistical notion of explaining variance. As already emphasized, variance is not the only measure of uncertainty, and moreover, variance is often not even a very useful measure of uncertainty if it is exceedance risks or tail probabilities that are of concern (which they usually are). Variance may partition, but uncertainty in the wider sense may not. Because the idea that partitioning of variance extends to other measures of uncertainty can lead to dangerous misconceptions, we call it the "anova myth".

Analysts often suggest it is an *empirical* fact that influential parameters are typically few in number in many practical situations in uncertainty analyses (Morgan and Henrion 1990; Cullen and Frey 1999). We would not dispute this idea. Indeed, we observe that it is often true as well. However, it seems clear that we cannot simply assume this will always be the case as Andres argues it is. The reason for the observation that only a few variables matter probably has to do with elementary combinatorial facts about the number of possible patterns with unequal distribution compared to the number of patterns with roughly equal distribution, rather than any mere definitional constraints.

7 Application to hard black box models

A black box model is one whose internal computational details are unknown to the analyst. Black box models are common in some engineering settings when, for example, the internal details should not be publicized for security, confidentiality, or intellectual property concerns, or when the internal details are available, but are so complex that projecting analyses through them is impractical. Some businesses and government agencies actually use legacy computer models for which the original source code has been lost. Sampling, in which the model is applied to a given set of input values and returns one or more output values, is often the only effective means to study black box models. This makes the study of computational black boxes much like empirical scientific inquiry of the nature world in that we can see the outcomes generated under particular input conditions, but cannot directly see into the inner workings that produced those results.

A *hard* black box model is one for which the number of samples is tightly constrained because of computational difficulty or other limits. The larger and more computer-intensive codes become, the harder the black box models are. Although the raw computational power available to analysts is still exponentially expanding, computer simulation codes are often developed with similarly increasing scientific and engineering complexity at the limits of practical computability. For practical purposes, we are interested in methods that can be applied to black box models in general, and especially

in any methods that can be applied to hard black box models for which relatively few samples will be available.

Various sampling-based schemes have been proposed that might be used to extend the new methods of sensitivity analysis based on Dempster-Shafer evidence theory and probability bounds analysis described in this report to black boxes. For example, Helton et al. (2004a,b) suggested decomposing the problem into a Cartesian product (à la Yager 1986) and solving the resulting matrix of interval problems by using black box sampling to estimate the ranges of these intervals with the observed ranges of the sample outputs. The approach is designed for problems involving Dempster-Shafer inputs, but it can be immediately applied to problems with p-boxes using the basic conversions described in Ferson et al. (2003). In principle, various ancillary strategies might be used to accelerate the convergence of this approach, such as methods that take account of overlap among the input intervals or the likely association of extreme values of the output variables with extreme values of the input variables, or employ strategic simplifications to reduce the dimensionality of the problem (Helton et al. 2006c). The approach was illustrated for a problem involving an algebraic expression in Helton et al. (2004b), and for a much more complex problem involving competing failure risks of strong and weak switches in Helton et al. (2004a).

Bruns et al. (2006a) described an alternative direct sampling approach called "optimized parameter sampling". This approach can be applied in situations where the inputs are "parameterized" p-boxes, which are essentially collections of distributions of a given shape (such as normal) specified by one or two parameters from within given intervals. In an outer sampling loop, distributions are selected from all the k input p-boxes by randomly picking the scalar parameter values from their respective intervals. For each collection of k (precise) probability distributions, sampling-based techniques are employed in an inner loop to solve the twin optimization problems of finding the upper and lower bounds on the expectation or any percentile of the result. Bruns et al. (2006a) illustrated the sampling strategy and evaluated its efficiency on a problem estimating the first passage time for a thermocouple temperature.

Bruns et al. (2006b) also described yet another direct sampling approach called p-box convolution sampling, although it can be applied immediately to Dempster-Shafer structures too. It involves taking random samples from each of the k inputs. A random sample from an uncertain number is the interval corresponding to the $(r\ 100)^{th}$ percentile where r is a random number uniformly distributed on the unit interval [0,1]. This generalizes the selection of a random value from a precise probability distribution (Ferson and Ginzburg 1995; Cooper et al. 1996). These k intervals are then projected through the black box model using sampling-based optimization techniques to find the largest and smallest output values given the input intervals, or the Cauchy-deviate sampling strategy described by Trejo and Kreinovich and (2001).

It does not seem possible to account for uncertainty about dependence among the inputs by generalizing the approach of Helton et al. (2004a,b) or either the optimized

parameter sampling* or p-box convolution sampling approaches of Bruns et al. (2006a,b). Even if it is possible to relax this ristriction, all of these direct sampling approaches will remain computationally intensive and would likely produce reliable results only when fairly many sample evaluations of the black box can be made. Despite their limitations and computational costs, these sampling approaches could make the new methods workable for a variety of problems in engineering.

These direct sampling methods permit the application of DST and PBA for sensitivity analysis in the context of epistemic uncertainty, including the use of the methods as sensitivity or robustness analyses of first-order probabilistic assessments as described in section 2, and the various kinds of pinching analyses described in section 4. The calculation of local sensitivity coefficients characterized by uncertain numbers (as described in section 3) for black box models based on a pure sampling approach would be possible if the approach of Uryasev (1994; 1995; 1997)—which computes local sensitivities of probabilities directly from Monte Carlo simulations without additional simulation runs—could somehow be generalized to handle epistemic uncertainty. If the details inside the black box are available in software source code (i.e., if the model is really a "crystal box" that can be scrutized but not changed), then the calculation of local sensitivity coefficients might also be accomplished with the use of automatic differentiation (http://www.mcs.anl.gov/Projects/autodiff; Griewank 1989; Fischer 1993; Korivi et al. 1994; Berz et al. 1996; Griewank 2000).

Aside from the direct sampling approaches described above, the general strategy of response surface modeling (Myers 1971; Morton 1983; Downing et al. 1985; Kleijnen 1992; Myers 1999; Myers et al. 2004) allows an indirect application of Dempster-Shafer theory and probability bounds analysis to black boxes that might often be useful and effective. Response surface modeling is widely employed in engineering to replace a black box that is too hard to study directly with a statistical model of the black box that is more amenable to detailed analysis. In principle, the response surface models can have any form, but usually a linear or low-order polynomial model is employed, which is often characterized as a "model of the model" in that it is a nakedly phenomenological model of a much richer, physics-based model.

Many analysts (e.g., Frey and Patil 2002) suggest that it will often be advantageous to limit the number of inputs that are included in the response surface model to those that are identified as the most important using some screening sensitivity analysis. Apart from the chicken-and-egg problem of having to decide what is important to a sensitivity analysis before one conducts a sensitivity analysis, there is a more fundamental objection

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*Bruns et al. (2006a) suggested that the optimized parameter sampling approach does not generalize to account for uncertainty about distribution family or to general Dempster-Shafer structures. However, this generality might be possible by devising a scheme for "sampling" discrete probability distributions from a non-parametric uncertain number. The theory of Chebyshev systems (Karlin and Studden 1966) suggests that the bounds on uncertain numbers are set by degenerate discrete probability distributions having mass only on a minimal number of points. It should be possible to generate example distributions that are consistent with given p-boxes or Dempster-Shafer structures.

to this suggestion: several individually unimportant variables may, in aggregate, be important. Although it may be reasonable to drop small-impact terms when trying to make an approximation, this is not a good idea when trying to *bound uncertainty*. In the context of epistemic uncertainty, it may be far more reasonable to simplify the problem in other ways, such as replacing a complex Dempster-Shafer or p-box representation of an input variable with its interval support. Such replacements cannot lead to underestimates of uncertainty, even if there are many of them. This strategy would therefore be preferable in many situations to simply omitting variables. On the other hand, the strategy cannot by itself overcome the problem of having very few samples.

Another important consideration also argues against omitting any of the inputs before computing the regression. In a reduced regression analysis, the regression coefficients cannot be directly interpreted as sensitivities associated with the terms of the regression. This is because regression coefficients can change, sometimes dramatically or even in sign, when the regression model is altered. When developing a response surface model, at least analysts know the correct inputs to use for their black-box model. (In this way they are better off than regression analysts in general who do not have such information.) Omitting some of the inputs to simplify the response surface is problematic because it creates the same disadvantage of varying regression coefficients for the response surface modeler.

The selection of the inputs to be used in sampling is an important consideration in response surface modeling, and this problem is treated in the broad statistical literature on sampling and experimental design. Typically, the inputs can be chosen by the analyst, although the design of inputs for sampling may occasionally not be under the analyst's control. This can happen when, for example, samples were obtained for other purposes (such as calibration) and additional sampling would be costly. When an analyst can specify the inputs for the samples, randomness of sample design is often a good strategy in many situations and usually simplifies statistical inferences, but various stratified sampling strategies such as Latin hypercube sampling or importance sampling may more commonly be preferable (Helton and Davis 2000a; 2002; 2003).

Perhaps even more critical than where the points will be is the issue of how many points there will be. If there are n input dimensions, one needs a minimum of n+1 sample points in general position to specify a linear model. Many more points would be necessary to specify a full quadratic or higher-order model. If there are more input dimensions than there are sample points, then the regression is underdetermined and cannot be performed by objective statistical methods. There are infinitely many planes that pass through two points. It may still be possible, nevertheless, to use response surfaces even in these extreme cases if the analyst can interject mechanistic knowledge of the physics of the underlying process to specify the model.

The difference between the original sample output and the output that would be predicted from the response surface model applied to the corresponding sample input is called the residual. The statistical fit of a response surface model and the normality of residuals can be studied using various standard well-known techniques such as the Kolmogorov-Smirnov, Anderson-Darling or chi-squared tests. For many hard black boxes, however, the goodness of fit of the response surface model is rarely an issue

because the number of available sample points is so few relative to the dimensionality of the model. If a response surface can be selected to pass through all the available points, the residuals are zero.

In applications where there are non-zero residuals, the uncertainty that they embody should not be neglected in the subsequent analysis. A response surface model fitted by least-squares regression, for example, will have a form like

$$y = \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_k x_k + \varepsilon$$

where the response variable y is explained as a sum of inputs x_i multiplied respectively by associated regression coefficients β_i, and an error term ε which is represented by a normal distribution having mean zero and standard deviation σ (which is essentially the standard deviation of the residuals). Typically, ε is assumed to be independent of the other terms in the regression model. This error term should be incorporated into any subsequent sensitivity or uncertainty analyses based on this response surface model. Failure to do so could clearly understate the true uncertainty.

Once a black box model is represented by a response surface model, the methods of uncertainty and sensitivity analysis outlined in the previous sections of this report can be directly applied, including ordinary Dempster-Shafer theory and probability bounds analysis and pinching analyses of various kinds. The computational difficulty associated with these applications may be fairly low if the response surface is a first-order linear model. If it includes repeated variables such as squares or higher powers or cross products representing interactions in addition to linear terms, then more careful strategies that account for the repeated variables will be needed to obtain best possible results. Strategies that may be useful in such cases are reviewed by Kreinovich et al. (2006).

There is a loss of guaranteed rigor in the use of a response surface model rather than the original model. This means that, even if the uncertainty of the input variables is surely captured by their uncertain number representations (Dempster-Shafer structures or p-boxes) and original model is an exact representation of the underlying process, the fitting of a response surface model is a statistical exercise and it may be imperfect. Indeed, it would be expected to be imperfect when there are few sample points available to inform the regression. We know of no method that would allow an analyst to rigorously propagate uncertainty through a black box model without assumptions that make the results contingent on the presumption that the response surface model is correct. The absence of the guarantee means that the uncertainty and sensitivity analyses of hard black box models will be approximate. Nevertheless, these approximations can often be good enough for use throughout engineering.

8 Conclusions and future directions

Although sensitivity analysis is universally recognized as crucial in planning strategies to manage risks of adverse events, as well as in designing further empirical study to improve risk estimates, comprehensive sensitivity analyses are not always

undertaken for probabilistic calculations because of the computational burden they entail. Many probabilistic assessments employ what-if sensitivity studies to explore the possible impact on the assessment results of varying the inputs. However, such studies are often difficult to conduct because of the large number of calculations that are required and because the universe of alternative models is ill-defined or difficult to sample. Although this approach can be informative, it is rarely comprehensive because, when there are multiple uncertainties at issue (as there usually are), the shear factorial problem of computing all of the possible combinations becomes prohibitive. Usually only a relatively tiny number of such analyses can be performed in practice. Yet when uncertainties are large and precise estimates of probabilities are impossible to obtain, characterizations of sensitivities become even more important to the decision making process. In many risk and safety assessments, DST and PBA can be used to automate such what-if sensitivity studies and vastly increase their comprehensiveness.

Sensitivity analysis can also be conducted within DST and PBA by hypothetically replacing an uncertain number (Dempster-Shafer structure or p-box) with a precise distribution, a zero-variance interval, or perhaps a scalar number to evaluate the potential reduction of uncertainty of the result under additional knowledge. These different kinds of pinchings enable analysts to clearly distinguish the various purposes of sensitivity analyses. Pinching away epistemic uncertainty is useful in planning future empirical effort. Pinching away aleatory uncertainty is useful in planning engineering interventions to control system outcomes. The dike revetment case study exemplified these different kinds of pinchings and produced results pointing to different variables as sensitive in different senses.

DST and PBA permit a comprehensive uncertainty analysis, and this fact obviates some of the complexity that attends traditional approaches to sensitivity analysis based on similar pinching ideas. For instance, when a variable is pinched to a point value in a Monte Carlo sensitivity study, the analyst usually wants to pinch to many possible values (according to their respective probabilities) and find the *average* effect of the pinching. This is called "freezing" the variable. The analog of freezing in PBA or DST would be to replace an uncertain number with many possible precise distributions and find the *envelope* of the results yielded under the various pinchings. But the original analysis already produced this envelope in the calculation for the baseline case. Whichever precise distribution the uncertain number is pinched to, the result of the pinched calculation is sure to lie within the original baseline case. This is true even if multiple variables are pinched simultaneously. Thus, sensitivity studies conducted on top of PBA or DST may not need to be as complicated as they are for traditional Monte Carlo simulations.

It might, nevertheless, be possible to conduct a sensitivity analysis within DST or PBA using averaging rather than enveloping, although doing so requires developing a generalization of the notion of 'average' that is meaningful in settings involving imprecise probabilities. In this case, the average of a class of distributions would be a set of values rather than a single scalar quantity. Hall (2006) explores this approach and shows that variance-based or entropy-based sensitivity indices can be extended to the

context of imprecise probabilities. He gives numerical examples of how they might be computed when the possible probability distributions are defined by interval parameters.

Much of the current literature on sensitivity analysis for probabilistic settings is devoted to the idea of variance decomposition. Variance is sometimes a reasonable measure of overall uncertainty for probability distributions, although it is hardly the only possible such measure. In the context of uncertain numbers, variance by itself seems to be a decidedly impoverished measure of overall uncertainty. To represent aleatory and epistemic uncertainty simultaneously, it may be necessary to introduce a bivariate measure that characterizes the epistemic and aleatory components separately. In this report, we have found the breadth (area between the upper and lower bounds of a p-box) and the interval estimate of variance to be an informative pair for describing uncertain numbers.

Sampling strategies may be useful in extending PBA and DST to black box models which can only be studied by sampling, but they are likely to require large numbers of samples to yield reliable results. Response surface models are likewise non-rigorous, especially when the sample sizes used to develop them are limited, but they may also be commonly useful in engineering practice. In constructing response surface models, it is important not to artificially restrict the number of inputs used in the regression analysis to ensure that uncertainty in all input variables can be propagated. It is also important to propagate the uncertainty associated with the residual or error term from the regression analysis through any subsequent uncertainty analyses.

A plethora of methods for sensitivity analysis have been proposed for probabilistic analyses. It is not entirely reasonable to compare these methods against each other as if they were exchangeable. They are appropriate for different kinds of problems, and they were designed to be convenient for different methodological approaches. Most importantly, they answer different questions. For example, the sampling-based methods of sensitivity analysis described by Helton et al. (2006b) are especially useful for tracking analyses that match which particular inputs correspond to outcomes of special interest. They are useful for answering questions such as what combination of extreme load and weather conditions will result in the collapse of a structure. The new methods of DST and PBA do not seem to be well suited for this purpose. On the other hand, the tracking analyses are not well suited to answering other important questions such as "how robust are the results of the assessment?" or "what variables should be studied to most quickly reduce the overall uncertainty?" for which DST and PBA are especially well suited. The fact that very different kinds of analyses for different purposes have been lumped together under the name "sensitivity analysis" has certainly led to some confusion. It would be useful to tease apart the various purposes, e.g., assessing robustness, planning empirical effort, planning engineering intervention, tracking causation, etc., in a comparison of methods.

8.1 Future directions

We recognize the following possible directions where future work would be useful.

1. Further research is needed to develop and assess practical computational strategies for DST and PBA for effective use with black box models. Especially

valuable would be methods that can yield reasonably conservative results even when very few evaluations of the model are possible.

2. Optimized parameter sampling which can project the uncertainty in parameterized p-boxes through black box models should be generalized to handle non-parametric p-boxes and Dempster-Shafer structures. This might be possible by appeal to the theory of Chebyshev systems (Karlin and Studden 1966).
3. Research is needed on the functions that might be possible and useful for measuring the overall uncertainty of uncertain numbers. The report's focus on the breadth (area between the bounds of the p-box) belies the fact that many other measures are possible, and some might be substantially better for particular purposes.
4. Computational methods are needed that can produce assured enclosures (rigorous bounds) for the reductions of uncertainty that are computed by pinching. The simulation methods used in this report are fast and probably often good approximations, but they lack the guarantee that accompanies most other calculations in DST and PBA.
5. It would be useful to establish that it is not only possible to use the new methods to compute sensitivities but also that it is indeed a good idea to do so. Further research is needed to compare the new methods with other available methods that might be used in the context of epistemic uncertainty. Criteria for a successful method of sensitivity analysis were discussed by Swartzman and Kaluzny (1987).
6. The method of Uryasev (1994; 1995; 1997), which can compute local sensitivities of probabilities directly from Monte Carlo simulations with negligible additional calculation and no additional simulation runs, should be extended to handle epistemic uncertainty in the form of intervals, p-boxes or Dempster-Shafer structures.

9 Glossary

automatic differentiation A computerized method for the numerical evaluation of the derivative of a function specified by a computer program at a specific point. It is distinguished from symbolic differentiation in which a function is symbolically manipulated to derive the derivative function for all point values, and also from numerical differentiation which is an approximate method for evaluating a derivative at a point based on discretized finite differences. Automatic differentiation is sometimes called algorithmic differentiation.

aleatory uncertainty The kind of uncertainty resulting from randomness or unpredictability due to stochasticity. Aleatory uncertainty is also known as variability, stochastic uncertainty, Type I or Type A uncertainty, irreducible uncertainty, or objective uncertainty.

almost surely A property holds almost surely if it holds always except possibly for a set of measure zero.

Bayesian sensitivity analysis A method to estimate the robustness of answers from a Bayesian analysis to uncertainty about the details and inputs of the analysis (Berger 1985; 1994). An answer is robust if it does not depend sensitively on the assumptions and inputs on which it is based. Bayesian sensitivity analysis is also called robust Bayes analysis.

best possible As tight as can be justified (said of bounds). An upper bound is best possible if is the smallest such bound possible, and a lower bound is best possible if it is the largest lower bound possible.

bound An upper bound of a set of real numbers is a real number that is greater than or equal to every number in the set. A lower bound is a number less than or equal to every number in the set. In this report, we also consider bounds on functions. These are not bounds on the range of the function, but rather bounds on the function for every function input. For instance, an upper bound on a function $F(x)$ is another function $B(x)$ such that $B(x) \geq F(x)$ for all values of x. $B(x)$ is a lower bound on the function if the inequality is reversed. If an upper bound cannot be any smaller, or a lower bound cannot be any larger, it is called a best possible bound.

CDF Cumulative distribution function (see distribution function).

convolution The mathematical operation which finds the distribution of a sum of random variables from the distributions of its addends. The term can be generalized to refer to differences, products, quotients, etc. It can also be generalized to refer to intervals, p-boxes and Dempster-Shafer structures as well as distributions.

core The set of possible values of an uncertain number about which information expressed in the uncertain number is totally vacuous. The core of a Dempster-Shafer structure is the intersection of all its focal elements. The core of a p-box is the region, if one exists, along the abscissa for which the upper bound of the p-box is one and lower bound is zero. The core of an interval is the interval itself. A

probability distribution does not have a core, except for the degenerate case of a Dirac delta distribution.

correlation The tendency of two paired variables to vary in the same direction. Correlation can be measured by several coefficients, conventionally constrained to the interval [−1, +1], such as Pearson correlation, Spearman correlation, Kendall correlation, among others. If used without qualification, correlation usually refers to Pearson correlation.

covariance The first product moment of two variables about their means. For random variable X and Y, their covariance is $\mathrm{cov}(X, Y) = \mathrm{E}((X - \mathrm{E}(X))\,(Y - \mathrm{E}(Y))) = \mathrm{E}(XY) - \mathrm{E}(X)\,\mathrm{E}(Y)$.

cumulative distribution function A distribution function.

Dempster-Shafer structure A set of focal elements (in this report, closed intervals of the real line), each of which is associated with a non-negative real values that sum to unity.

Dempster-Shafer theory A variant of probability theory in which the elements of the probability space to which nonzero mass is attributed, called focal elements, are not singletons but rather sets which represent the indistinguishabilty of alternatives within bodies of evidence. Dempster-Shafer theory is sometimes called evidence theory.

dependence The relationship between events or between random variables. If one event (random variable) is unrelated to another event (random number), they are said to be independent. Otherwise, they are said to be dependent.

dependency bounds Bounds on a sum or other arithmetic function that arise when no assumption is made about the inter-variable dependence between the addend distributions or input distributions of the function.

Dirac delta distribution A degenerate kind of probability distribution that corresponds to a constant scalar quantity.

dispersive Monte Carlo simulation A Monte Carlo simulation in which unknown correlations are set to their most extreme plausible values in order to obtain results that conservatively estimate variances and tail probabilities.

distribution function The function F associated with some variate that describes the probability $F(X)$ that the variate will take on a value not greater than X. The distribution function associated with a data set of scalar values describes the probability $F(X)$ that a value selected at random (i.e., uniformly and independently) from the data values will have a value less than X. Also known as a cumulative distribution function.

DST Dempster-Shafer theory.

epistemic uncertainty The kind of uncertainty arising from imperfect knowledge rather than variability. Epistemic uncertainty is also known as incertitude, ignorance,

subjective uncertainty, Type II or Type B uncertainty, reducible uncertainty, and state-of-knowledge uncertainty.

event A subset of the sample space, which is the set of all possible outcomes of a random experiment. If the outcome of the random experiment is a member of an event, then the event is said to have occurred. In probability theory, an event is a collection of outcomes for which a probability has been assigned.

focal element A set (in this report, a closed interval of the real line) associated with a nonzero mass as a part of a Dempster-Shafer structure.

Fréchet case The strategy of making no assumption about dependence.

Fréchet bounds Bounds on a joint distribution $H(x,y)$, specified by having marginal distributions $F(x)$ and $G(y)$, given by

$$\max\big(F(x)+G(y)-1,\ 0\big) \le H(x,y) \le \min\big(F(x),\ G(y)\big).$$

These bounds are also known as the Fréchet-Hoeffding limits (Fréchet 1951; Hoeffding 1940). They are the distributional analogs of the bounds in the Fréchet inequalities.

Fréchet inequalities Inequalities due to Fréchet (1935) on the probabilities of conjunctions and disjunctions of events A_i given by

$$\max(0,\ a_1 + a_2 + \ldots + a_n - (n-1)) \le \mathrm{P}(A_1 \,\&\, A_2 \,\&\, \ldots \,\&\, A_n) \le \min(a_1, a_2, \ldots, a_n),$$

$$\max(a_1, a_2, \ldots, a_n) \le \mathrm{P}(A_1 \vee A_2 \vee \ldots \vee A_n) \le \min(1,\ a_1 + a_2 + \ldots + a_n),$$

where $a_i = \mathrm{P}(A_i)$.

imprecise probabilities The subject of any of several theories involving models of uncertainty that do not assume a unique underlying probability distribution, but instead correspond to a set of probability distributions (Couso et al. 2000). The lower probability $\underline{\mathrm{P}}(A)$ for event A is the maximum rate one would be willing to pay for a gamble that pays 1 unit of utility if A occurs and nothing otherwise. The upper probability $\overline{\mathrm{P}}(A)$ for event A is $1-\underline{\mathrm{P}}(\text{not } A)$, i.e., one minus the lower probability of A not occurring. An imprecise probability arises when one's lower probability for an event is strictly smaller than one's upper probability for the same event (Walley 1991). Theories of imprecise probabilities are often expressed in terms of a lower probability measure giving the lower probability for every possible event from some universal set, or in terms of closed convex sets of probability distributions. Interval probabilities, Dempster-Shafer structures and probability boxes can be regarded as special-cases of imprecise probabilities.

incertitude The kind of uncertainty arising from imperfect knowledge. Incertitude is also known as epistemic uncertainty, ignorance, subjective uncertainty, Type II or Type B uncertainty, reducible uncertainty, and state-of-knowledge uncertainty.

infimum The greatest lower bound of a set of values. When the set consists of a finite collection of closed intervals, the infimum value is the same as the minimum value.

interval The set of all real numbers lying between two fixed numbers called the endpoints of the interval. In this report, intervals are always closed so that the endpoints are considered part of the set.

inverse function For a function $y = F(x)$, an inverse function F^{-1} takes y-values in the range of the function F to x-values in the domain of F in such a way that $F^{-1}(F(x)) = x$ and $F(F^{-1}(y)) = y$. For instance, if $F(x)$ is the distribution function for a random variable X giving the probability associated with the event $X \leq x$, then the inverse function $F^{-1}(p)$ is the value of x associated with any value p. An inverse function does not necessarily exist for any function, but any one-to-one function will have an inverse.

mean The probability-weighted average of a set of values or a probability distribution. The mean is also called the expected value or the expectation of a random variable. It is the first moment of a probability distribution.

Monte Carlo simulation A method of calculating functions of probability distributions by repeatedly sampling random values from those distributions and forming an empirical distribution function of the results.

NP-hard Lacking a general polynomial-time algorithm for a solution. NP-hard problems grow exponentially in complexity with sample size or other feature. Also called NP-complete.

outward-directed rounding Rounding an upper bound upward and a lower bound downward so as to maintain rigor of the bounds in reduced-precision expressions.

p-box A probability box.

PBA Probability bounds analysis.

probability bounds analysis An analysis or calculation involving interval probabilities or probability boxes.

probability box A class of distribution functions $F(x)$ specified by a bounding pair of distribution functions $\underline{F}(x)$ and $\overline{F}(x)$ such that $\underline{F}(x) \leq F(x) \leq \overline{F}(x)$ for all x values.

quantile A number that divides the range of a set of data or a distribution such that a specified fraction of the data or distribution lies below this number.

random variable A variable quantity whose values are distributed according to a probability distribution. If the potential values of the random variable are a finite or countable set, the random variable is said to be discrete. For a discrete random variable, each potential value has an associated probability between zero and one, and the sum of all of these probabilities is one. If the random variable can take on any value in some interval of the real line (or any rational value within some interval), it is called a continuous random variable.

real number A real number is an element from the real line consisting of positive and negative integers, rational numbers, irrationals and transcendental numbers. A real number is a rational number or the limit of a sequence of rational numbers. Real numbers are sometimes called scalars.

repeated variables problem The computational difficulty of evaluating accurate or best possible results from a mathematical expression involving multiple instances of uncertain variables or parameters that arises because the dependence between the repetitions is not taken into account in the evaluation. For example, given an interval $A=[a_1,a_2]$, the evaluation of the $A - A = [a_1 - a_2, a_2 - a_1]$, is an interval that only straddles zero, rather than being identically zero as one might expect. The same problem can also arise in Monte Carlo simulations whenever the perfect correlation between multiple instances of a variable within an expression is not recognized, as can occur when the evaluation of the expression is undertaken piecewise rather than all at once, but, in the case of Monte Carlo simulations, the uncertainty is usually underestimated. Various techniques and strategies are available to overcome the repeated variable problem.

rigorous Exact or sure, as opposed to merely approximate. Usually said of bounds which can be rigorous without being best possible.

robust Bayes analysis Bayesian sensitivity analysis.

sensitivity analysis A method that assesses the sensitivity of a model's output to changes in one or more of its input parameters. If small changes in an input parameter result in relative large changes in a model's output, the model is said to be sensitive to the parameter.

support The subset of the domain of a distribution function over which the function is neither perfectly zero nor perfectly one.

supremum The least upper bound of a set of values. When the set consists of a finite collection of closed intervals, the supremum value is the same as the maximum value.

total probability The probability of a single event.

two-dimensional Monte Carlo A kind of nested Monte Carlo simulation in which distributions representing both epistemic uncertainty and aleatory uncertainty are combined together. Typically, the outer loop selects random values for the parameters specifying the distributions used in an inner loop to represent aleatory uncertainty. This approach is also called second-order Monte Carlo simulation.

uncertain number A numerical quantity or distribution about which there is uncertainty. Uncertain numbers include intervals, probability distributions, probability boxes, Dempster-Shafer structures as special cases. Uncertain numbers also include scalars (real numbers) as degenerate special cases.

uncertainty The absence of perfectly detailed knowledge. Uncertainty includes epistemic uncertainty (the exact value is not known) and aleatory uncertainty (the value is changing). Uncertainty may also include other forms such as vagueness, ambiguity and fuzziness (in the sense of border-line cases).

uncertainty analysis A variety of calculations or techniques for quantitative or qualitative assessments explicitly involving uncertainty, often including listing of the sources of uncertainty impinging on a problem or quantity (uncertainty

budgets), characterization of the uncertainty about a numerical quantity, propagation of uncertainty (uncertainty quantification) through a model, or assessment of the relative importance of the uncertainties of various inputs of a model in terms of their contributions to its output's uncertainty.

uncertainty quantification Calculation of the overall uncertainty in a model's output due to the uncertainties in its various inputs. Also called uncertainty propagation.

variability The fluctuation or variation due to randomness or stochasticity. Variability is also called aleatory uncertainty, stochastic uncertainty, Type 1 or Type A uncertainty, irreducible uncertainty and objective uncertainty.

variance A measure of the variability in the values of a random variable defined as the mean of the squared difference between the random variable and its mean value.

variance-based sensitivity analysis An approach to sensitivity analysis for probabilistic computations that uses variance as a measure of the overall uncertainty of a probability distribution.

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San Antonio, TX 78284-0510

Bilal Ayyub (2)
Department of Civil Engineering
University of Maryland
College Park, MD 20742-3021

Ivo Babuska
TICAM
Mail Code C0200
University of Texas at Austin
Austin, TX 78712-1085

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Wright-Patterson AFB, OH 45433

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Duke University
Box 90251
Durham, NC 27708-0251

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Laboratory for Computational Physics and Fluid Dynamics
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James C. Cavendish
General Motors R&D Center
Mail Code 480-106-359
30500 Mound Road
Box 9055
Warren, MI 48090-9055

Chun-Hung Chen (2)
Department of Systems Engineering &
Operations Research
George Mason University
4400 University Drive, MS 4A6
Fairfax, VA 22030

Wei Chen (2)
Department of Mechanical Engineering
Northwestern University
2145 Sheridan Road, Tech B224
Evanston, IL 60208-3111

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MC 4040
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Boeing-Phantom Works
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St. Louis, MO 63166-0516

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University of Illinois at Chicago
Chemical Engineering Dept.
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254C Kaprielian Hall
Dept. of Civil Engineering
3620 S. Vermont Ave.
University of Southern California
Los Angles, CA 90089-2531

Mike Giltrud
Defense Threat Reduction Agency
DTRA/CPWS
6801 Telegraph Road
Alexandria, VA 22310-3398

James Glimm (2)
Dept. of Applied Math & Statistics
P138A
State University of New York
Stony Brook, NY 11794-3600

Ramana Grandhi (2)
Dept. of Mechanical and Materials Engineering
3640 Colonel Glenn Hwy.
Dayton, OH 45435-0001

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The National Institute of Aerospace
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Dept. of Aerospace and Mechanical
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Dept. of Civil Engineering
& Engineering Mechanics
University of Arizona
Tucson, AZ 85721

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ACTA
2790 Skypark Dr., Suite 310
Torrance, CA 90505-5345

George Hazelrigg
Design, Manufacturing & Innovation
4201 Wilson Boulevard, Room 508N
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Mechanical Engineering Dept.
New Mexico State University
P. O. Box 30001/Dept. 3450
Las Cruces, NM 88003-8001

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102 Donner Drive
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P. O. Drawer 28510
San Antonio, TX 78284-0510

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Northrop Grumman Information Technology
222 West Sixth St.
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Sience and Technology Policy Intern
Board of Mathematical Sciences and Applications
500 5th Street, NW
Washington, DC 20001

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Research Institutes Building
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George Karniadakis (2)
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192 George St., Box F
Providence, RI 02912

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2145 Sheridan Road
Evanston, IL 60108-3111

Robert Lust
General Motors, R&D and Planning
MC 480-106-256
30500 Mound Road
Warren, MI 48090-9055

Sankaran Mahadevan (2)
Dept. of Civil &
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Vanderbilt University
Box 6077, Station B
Nashville, TN 37235

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Institute for Defense Analysis
Operational Evaluation Division
4850 Mark Center Drive
Alexandria VA 22311-1882

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NDM Solutions
1420 Aldenham Lane
Reston, VA 20190-3901

Gregory McRae (2)
Dept. of Chemical Engineering
Massachusetts Institute of Technology
Cambridge, MA 02139

Michael Mendenhall (2)
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Mountain View, CA 94043

John G. Michopoulos
Naval Research Laboratory,
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Computational Mutliphysics Systems
Lab
Washington DC 20375, USA

Sue Minkoff
Dept. of Mathematics and Statistics
University of Maryland
1000 Hilltop Circle
Baltimore, MD 21250

Max Morris (2)
Department of Statistics
Iowa State University
304A Snedecor-Hall
Ames, IW 50011-1210

Rafi Muhanna
Regional Engineering Program
Georgia Tech
210 Technology Circle
Savannah, GA 31407-3039

R. Namburu
U.S. Army Research Laboratory
AMSRL-CI-H
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Albuquerque, NM 87110

Efstratios Nikolaidis (2)
MIME Dept.
4035 Nitschke Hall
University of Toledo
Toledo, OH 43606-3390

Tinsley Oden (2)
TICAM
Mail Code C0200
University of Texas at Austin
Austin, TX 78712-1085

Michael Ortiz (2)
Graduate Aeronautical Laboratories
California Institute of Technology
1200 E. California Blvd./MS 105-50
Pasadena, CA 91125

Dale K. Pace
4206 Southfield Rd
Ellicott City, MD 21042-5906

Alex Pang
Computer Science Department
University of California
Santa Cruz, CA 95064

Chris Paredis
School of Mechanical Engineering
Georgia Institute of Technology
813 Ferst Drive, MARC Rm. 256
Atlanta, GA 30332-0405

Chris L. Pettit
Aerospace Engineering Dept.
MS-11B
590 Holloway Rd.
Annapolis, MD 21402

Allan Pifko
2 George Court
Melville, NY 11747

Cary Presser (2)
Process Measurements Div.
National Institute of Standards
and Technology
Bldg. 221, Room B312
Gaithersburg, MD 20899

Gerald R. Prichard
Principal Systems Analyst
Dynetics, Inc.
1000 Explorer Blvd.
Huntsville, AL 35806

Thomas A. Pucik
Pucik Consulting Services
13243 Warren Avenue
Los Angles, CA 90066-1750

W. Rafaniello
DOW Chemical Company
1776 Building
Midland, MI 48674

J. N. Reddy
Dept. of Mechanical Engineering
Texas A&M University
ENPH Building, Room 210
College Station, TX 77843-3123

John Renaud (2)
Dept. of Aerospace & Mechanical Engr.
University of Notre Dame
Notre Dame, IN 46556

Grant Reinman
Pratt & Whitney
400 Main Street, M/S 162-01
East Hartford, CT 06108

Tim Ross (2)
Dept. of Civil Engineering
University of New Mexico
Albuquerque, NM 87131

Chris Roy (2)
Dept. of Aerospace Engineering
211 Aerospace Engineering Bldg.
Auburn University, AL 36849-5338

J. Sacks
Inst. of Statistics and Decision Science
Duke University
Box 90251
Durham, NC 27708-0251

Sunil Saigal (2)
Carnegie Mellon University
Department of Civil and
Environmental Engineering
Pittsburgh, PA 15213

Larry Sanders
DTRA/ASC
8725 John J. Kingman Rd
MS 6201
Ft. Belvoir, VA 22060-6201

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and Geological Sciences
University of Notre Dame
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G. R. Srinivasan
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1111 Lockheed Martin Way
Sunnyvale, CA 94089

D. E. Stevenson (2)
Computer Science Department
Clemson University
442 Edwards Hall, Box 341906
Clemson, SC 29631-1906

Ben Thacker
Southwest Research Institute
6220 Culebra Road
P. O. Drawer 28510
San Antonio, TX 78284-0510

Fulvio Tonon (2)
Dept. of Civil Engineering
University of Texas at Austin
1 University Station C1792
Austin, TX 78712-0280

Robert W. Walters (2)
Aerospace and Ocean Engineering
Virginia Tech
215 Randolph Hall, MS 203
Blacksburg, VA 24061-0203

Leonard Wesley
Intellex Inc.
5932 Killarney Circle
San Jose, CA 95138

Justin Y-T Wu
8540 Colonnade Center Drive, Ste 301
Raleigh, NC 27615

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Ford Research Laboratory
MD2115-SRL
P.O.Box 2053
Dearborn, MI 4812

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DOD/DMSO
Technical Director for VV&A
1901 N. Beauregard St., Suite 504
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Via Eurialo 28, Int. 13
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ITALY

D. Thunnissen
School of Mech. And Aerospace Engineering
Nanyang Technical University
50 Mamuamg Ave.
SINGAPORE 639798

Malcolm Wallace
Computational Dynamics Ltd.
200 Shepherds Bush Road
London W6 7NY
UNITED KINGDOM

Department of Energy Laboratories

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Mail Station 5000, P.O. Box 1663
Los Alamos, NM 87545
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